

A CHARACTERIZATION OF THE WAVE FRONT SET DEFINED BY THE ITERATES OF AN OPERATOR WITH CONSTANT COEFFICIENTS

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ABSTRACT. We characterize the wave front set $\text{WF}_*^P(u)$ with respect to the iterates of a linear partial differential operator with constant coefficients of a classical distribution $u \in \mathcal{D}'(\Omega)$, Ω an open subset in \mathbb{R}^n . We use recent Paley-Wiener theorems for generalized ultradifferentiable classes in the sense of Braun, Meise and Taylor. We also give several examples and applications to the regularity of operators with variable coefficients and constant strength. Finally, we construct a distribution with prescribed wave front set of this type.

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1. Introduction

We introduced in [3] the wave front set $\text{WF}_*^P(u)$, for a classical distribution $u \in \mathcal{D}'(\Omega)$ defined on a open set Ω of \mathbb{R}^n , with respect to the iterates of a hypoelliptic linear partial differential operator P with constant coefficients for ultradifferentiable classes in the sense of Braun, Meise and Taylor [7]. We established in [3] a microlocal regularity theorem for this wave front set and we studied the product of ultradifferentiable functions defined in the usual way with the ones defined by iterates. As a consequence we obtained a partial result related to the construction of distributions with prescribed wave front sets. Here, we describe more precisely the behaviour of the set $\text{WF}_*^P(u)$ and complete the previous result about prescribed singularities. Finally, we give some applications to the ω -micro-regularity of linear partial differential operators with variable coefficients and constant strength.

The problem of iterates begins mainly when Komatsu [16] in 1960s characterized analytic functions f in terms of the behaviour of successive iterates $P(D)^j f$ of a partial differential elliptic operator $P(D)$ with constant coefficients, proving that a C^∞ function f is real analytic in Ω if and only if for every compact set $K \subset\subset \Omega$ there is a constant $C > 0$ such that

$$\|P(D)^j f\|_{2,K} \leq C^{j+1} (j!)^m,$$

where m is the order of the operator and $\|\cdot\|_{2,K}$ is the L^2 norm on K . This result was generalized for elliptic operators with variable analytic coefficients by Kotake and Narasimhan [17, Theorem 1]. Later this result was extended to the setting of Gevrey functions by Newberger and Zielezny [23] and completely characterized by Métivier [22] (see also [26]). Spaces of Gevrey type given by the iterates of a differential operator are called *generalized Gevrey classes* and were used by Langenbruch [18, 19, 20, 21] for different purposes. For more references about generalized Gevrey classes and the microlocal version of the problem see [3].

More recently, Juan Huguet [14] extended the results of Komatsu [16], Newberger and Zielezny [23] and Métivier [22] to the setting of non-quasianalytic classes in the sense of Braun, Meise and Taylor [7]. In [14], Juan Huguet introduced the generalized spaces of ultradifferentiable functions $\mathcal{E}_*^P(\Omega)$ on an open subset Ω of \mathbb{R}^n for a fixed linear partial differential operator P with constant coefficients, and proved that these spaces are complete if and only if P is hypoelliptic. Moreover Juan Huguet showed that, in this case, the spaces are nuclear. Later, the same author in [15] established a Paley-Wiener theorem for the classes $\mathcal{E}_*^P(\Omega)$, again under the hypothesis of the hypoellipticity of P .

In order to remove the assumption on the hypoellipticity of the operator, we considered in [2] a different setting of ultradifferentiable functions, following the ideas of [4].

The structure of the paper is as follows. We begin in Section 2 with some notation and preliminaries. First, we introduce the classes of ultradifferentiable functions. Then we complete some partially known results on linear partial differential operators with constant coefficients regarding ω -regularity that we will use in the last section. In Section 3 we use Paley-Wiener theorems in [15] to characterize the wave front set $\text{WF}_*^P(u)$ introduced in [3] (see Corollaries 3.14 and 3.15). The main tools to establish this characterization are [3, Proposition 17], in which we proved that the product of a suitable Gevrey function and a function in $\mathcal{E}_*^P(\Omega)$ is still in $\mathcal{E}_*^P(\Omega)$ (observe that $\mathcal{E}_*^P(\Omega)$ is not an algebra for pointwise multiplication in general), and the application of pseudodifferential operators defined by symbols supported in a given cone to the description of the wave front set in Theorem 3.11 (see [25, Proposition 3.4.4] for the corresponding result in Gevrey classes). In the last section, Section 4, we give some applications and examples, in particular to operators with variable coefficients and constant strength. For this purpose we employ some known results on ω -micro-regularity of operators with constant strength (see [8, 9]).

2. Notation and preliminaries

Let us recall from [7] the definitions of weight functions ω and of the spaces of ultradifferentiable functions of Beurling and Roumieu type:

Definition 2.1. *A non-quasianalytic weight function is a continuous increasing function $\omega : [0, +\infty[\rightarrow [0, +\infty[$ with the following properties:*

- (α) $\exists L > 0$ s.t. $\omega(2t) \leq L(\omega(t) + 1) \quad \forall t \geq 0$;
- (β) $\int_1^{+\infty} \frac{\omega(t)}{t^2} dt < +\infty$,
- (γ) $\log(t) = o(\omega(t))$ as $t \rightarrow +\infty$;
- (δ) $\varphi_\omega : t \mapsto \omega(e^t)$ is convex.

Normally, we will denote φ_ω simply by φ .

For a weight function ω we define $\bar{\omega} : \mathbb{C}^n \rightarrow [0, +\infty[$ by $\bar{\omega}(z) := \omega(|z|)$ and again we denote this function by ω .

The *Young conjugate* $\varphi^* : [0, +\infty[\rightarrow [0, +\infty[$ is defined by

$$\varphi^*(s) := \sup_{t \geq 0} \{st - \varphi(t)\}.$$

There is no loss of generality to assume that ω vanishes on $[0, 1]$. Then φ^* has only non-negative values, it is convex, $\varphi^*(t)/t$ is increasing and tends to ∞ as $t \rightarrow \infty$, and $\varphi^{**} = \varphi$.

Example 2.2. The following functions are, after a change in some interval $[0, M]$, examples of weight functions:

- (i) $\omega(t) = t^d$ for $0 < d < 1$.
- (ii) $\omega(t) = (\log(1+t))^s$, $s > 1$.
- (iii) $\omega(t) = t(\log(e+t))^{-\beta}$, $\beta > 1$.
- (iv) $\omega(t) = \exp(\beta(\log(1+t))^\alpha)$, $0 < \alpha < 1$.

In what follows, Ω denotes an arbitrary subset of \mathbb{R}^n and $K \subset\subset \Omega$ means that K is a compact subset in Ω .

Definition 2.3. Let ω be a weight function.

(a) For a compact subset K in \mathbb{R}^n which coincides with the closure of its interior and $\lambda > 0$, we define the seminorm

$$p_{K,\lambda}(f) := \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^n} \left| f^{(\alpha)}(x) \right| \exp \left(-\lambda \varphi^* \left(\frac{|\alpha|}{\lambda} \right) \right),$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and set

$$\mathcal{E}_\omega^\lambda(K) := \{f \in C^\infty(K) : p_{K,\lambda}(f) < \infty\},$$

which is a Banach space endowed with the $p_{K,\lambda}(\cdot)$ -topology.

(b) For an open subset Ω in \mathbb{R}^n , the class of ω -ultradifferentiable functions of Beurling type is defined by

$$\mathcal{E}_{(\omega)}(\Omega) := \{f \in C^\infty(\Omega) : p_{K,\lambda}(f) < \infty, \text{ for every } K \subset\subset \Omega \text{ and every } \lambda > 0\}.$$

The topology of this space is

$$\mathcal{E}_{(\omega)}(\Omega) = \proj_{\substack{K \subset\subset \Omega \\ \lambda > 0}} \proj_{\lambda > 0} \mathcal{E}_\omega^\lambda(K),$$

and one can show that $\mathcal{E}_{(\omega)}(\Omega)$ is a Fréchet space.

(c) For a compact subset K in \mathbb{R}^n which coincides with the closure of its interior and $\lambda > 0$, set

$$\mathcal{E}_{\{\omega\}}(K) = \{f \in C^\infty(K) : \text{there exists } m \in \mathbb{N} \text{ such that } p_{K, \frac{1}{m}}(f) < \infty\},$$

This space is the strong dual of a nuclear Fréchet space (i.e., a (DFN)-space) if it is endowed with its natural inductive limit topology, that is,

$$\mathcal{E}_{\{\omega\}}(K) = \varinjlim_{m \in \mathbb{N}} \mathcal{E}_m^{\frac{1}{m}}(K).$$

(d) For an open subset Ω in \mathbb{R}^n , the class of ω -ultradifferentiable functions of Roumieu type is defined by:

$$\mathcal{E}_{\{\omega\}}(\Omega) := \{f \in C^\infty(\Omega) : \forall K \subset\subset \Omega \exists \lambda > 0 \text{ such that } p_{K,\lambda}(f) < \infty\}.$$

Its topology is the following

$$\mathcal{E}_{\{\omega\}}(\Omega) = \proj_{K \subset\subset \Omega} \mathcal{E}_{\{\omega\}}(K),$$

that is, it is endowed with the topology of the projective limit of the spaces $\mathcal{E}_{\{\omega\}}(K)$ when K runs the compact subsets of Ω . This is a complete PLS-space, that is, a complete space which is a projective limit of LB-spaces (i.e., a countable inductive limit of Banach spaces) with compact linking maps in the (LB)-steps. Moreover, $\mathcal{E}_{\{\omega\}}(\Omega)$ is also a nuclear and reflexive locally convex space. In particular, $\mathcal{E}_{\{\omega\}}(\Omega)$ is an ultrabornological (hence barrelled and bornological) space.

The elements of $\mathcal{E}_{(\omega)}(\Omega)$ (resp. $\mathcal{E}_{\{\omega\}}(\Omega)$) are called ultradifferentiable functions of Beurling type (resp. Roumieu type) in Ω .

In the case that $\omega(t) := t^d$ ($0 < d < 1$), the corresponding Roumieu class is the Gevrey class with exponent $1/d$. In the limit case $d = 1$, not included in our setting, the corresponding Roumieu class $\mathcal{E}_{\{\omega\}}(\Omega)$ is the space of real analytic functions on Ω whereas the Beurling class $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ gives the entire functions.

If a statement holds in the Beurling and the Roumieu case then we will use the notation $\mathcal{E}_*(\Omega)$. It means that in all cases $*$ can be replaced either by (ω) or $\{\omega\}$.

For a compact set K in \mathbb{R}^n , define

$$\mathcal{D}_*(K) := \{f \in \mathcal{E}_*(\mathbb{R}^n) : \text{supp } f \subset K\},$$

endowed with the induced topology. For an open set Ω in \mathbb{R}^n , define

$$\mathcal{D}_*(\Omega) := \varinjlim_{K \subset\subset \Omega} \mathcal{D}_*(K).$$

Following [14], we consider smooth functions in an open set Ω such that there exists $C > 0$ verifying for each $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$,

$$\|P^j(D)f\|_{2,K} \leq C \exp\left(\lambda \varphi^*\left(\frac{jm}{\lambda}\right)\right),$$

where K is a compact subset in Ω , $\|\cdot\|_{2,K}$ denotes the L^2 -norm on K and $P^j(D)$ is the j -th iterate of the partial differential operator $P(D)$ of order m , i.e.,

$$P^j(D) = P(D) \underbrace{\circ \cdots \circ}_j P(D).$$

If $j = 0$, then $P^0(D)f = f$.

Given a polynomial $P \in \mathbb{C}[z_1, \dots, z_n]$ of degree m , $P(z) = \sum_{|\alpha| \leq m} a_\alpha z^\alpha$, the partial differential operator $P(D)$ is defined as $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$, where $D = \frac{1}{i} \partial$.

The spaces of ultradifferentiable functions with respect to the successive iterates of P are defined as follows.

Let ω be a weight function. Given a polynomial P , an open set Ω of \mathbb{R}^n , a compact subset $K \subset\subset \Omega$ and $\lambda > 0$, we define the seminorm

$$(2.1) \quad \|f\|_{K,\lambda} := \sup_{j \in \mathbb{N}_0} \|P^j(D)f\|_{2,K} \exp\left(-\lambda \varphi^*\left(\frac{jm}{\lambda}\right)\right)$$

and set

$$\mathcal{E}_{P,\omega}^\lambda(K) = \{f \in \mathcal{C}^\infty(K) : \|f\|_{K,\lambda} < +\infty\}.$$

It is a normed space endowed with the $\|\cdot\|_{K,\lambda}$ -norm.

The space of *ultradifferentiable functions of Beurling type with respect to the iterates of P* is:

$$\mathcal{E}_{(\omega)}^P(\Omega) = \{f \in \mathcal{C}^\infty(\Omega) : \|f\|_{K,\lambda} < +\infty \text{ for each } K \subset\subset \Omega \text{ and } \lambda > 0\},$$

endowed with the topology given by

$$\mathcal{E}_{(\omega)}^P(\Omega) := \proj_{K \subset\subset \Omega} \proj_{\lambda > 0} \mathcal{E}_{P,\omega}^\lambda(K).$$

If $\{K_n\}_{n \in \mathbb{N}}$ is a compact exhaustion of Ω we have

$$\mathcal{E}_{(\omega)}^P(\Omega) = \proj_{n \in \mathbb{N}} \proj_{k \in \mathbb{N}} \mathcal{E}_{P,\omega}^k(K_n) = \proj_{n \in \mathbb{N}} \mathcal{E}_{P,\omega}^n(K_n).$$

This is a metrizable locally convex topology defined by the fundamental system of seminorms $\{\|\cdot\|_{K_n,n}\}_{n \in \mathbb{N}}$.

The space of *ultradifferentiable functions of Roumieu type with respect to the iterates of P* is defined by:

$$\mathcal{E}_{\{\omega\}}^P(\Omega) = \{f \in \mathcal{C}^\infty(\Omega) : \forall K \subset\subset \Omega \exists \lambda > 0 \text{ such that } \|f\|_{K,\lambda} < +\infty\}.$$

Its topology is defined by

$$\mathcal{E}_{\{\omega\}}^P(\Omega) := \proj_{K \subset\subset \Omega} \ind_{\lambda > 0} \mathcal{E}_{P,\omega}^\lambda(K).$$

As in the Gevrey case, we call these classes *generalized non-quasianalytic classes*. We observe that in comparison with the spaces of generalized non-quasianalytic classes as defined in [14] we add here m as a factor inside φ^* in (2.1), where m is the order of the operator P , which does not change the properties of the classes and will simplify the notation in the following.

The inclusion map $\mathcal{E}_*(\Omega) \hookrightarrow \mathcal{E}_*^P(\Omega)$ is continuous (see [14, Theorem 4.1]). The space $\mathcal{E}_*^P(\Omega)$ is complete if and only if P is hypoelliptic (see [14, Theorem 3.3]). Moreover, under a mild condition on ω introduced by Bonet, Meise and Melikhov [6], $\mathcal{E}_*^P(\Omega)$ coincides with the class of ultradifferentiable functions $\mathcal{E}_*(\Omega)$ if and only if P is elliptic (see [14, Theorem 4.12]).

We denote by

$$\widehat{f}(\xi) := \int e^{-i\langle x, \xi \rangle} f(x) dx$$

the classical Fourier transform $\mathcal{F}(f)$.

Now, let $P(D) = \sum_{|\alpha| \leq m} c_\alpha D^\alpha$ be a linear partial differential operator with constant coefficients, where $D = -i\partial$. We recall the notion of hypoellipticity in the C^∞ class: $P(D)$ is hypoelliptic in $\Omega \subseteq \mathbb{R}^n$ if $P(D)u \in C^\infty(\Omega)$ implies $u \in C^\infty(\Omega)$. In this case we also say that the polynomial $P(\xi) = \sum_{|\alpha| \leq m} c_\alpha \xi^\alpha$ is hypoelliptic.

We set

$$V = V(P) := \{\zeta \in \mathbb{C}^n : P(\zeta) = 0\}$$

and consider the distance from $\xi \in \mathbb{R}^n$ to V :

$$d(\xi) := \inf_{\zeta \in V} |\xi - \zeta|, \quad \xi \in \mathbb{R}^n.$$

From [13, Thms 11.1.1 and 11.1.3] (see also [25, Prop. 2.2.1]) we recall the following characterization of hypoellipticity, that will be useful in the following:

Theorem 2.4. *Let $P(D)$ be a linear partial differential operator with constant coefficients. The following properties are equivalent for P to be hypoelliptic:*

- (1) *For every open set $\Omega \subseteq \mathbb{R}^n$ and $u \in \mathcal{D}'(\Omega)$*

$$\text{WF}(u) = \text{WF}(P(D)u).$$

- (2) *For every open set $\Omega \subseteq \mathbb{R}^n$ and $u \in \mathcal{D}'(\Omega)$*

$$\text{sing supp } u = \text{sing supp } P(D)u.$$

- (3) *P is homogeneous hypoelliptic, i.e. if Ω is open in \mathbb{R}^n and $u \in \mathcal{D}'(\Omega)$ then $P(D)u = 0$ implies $u \in C^\infty(\Omega)$.*

- (4) $\lim_{\substack{\xi \in \mathbb{R}^n \\ |\xi| \rightarrow +\infty}} \frac{D^\alpha P(\xi)}{P(\xi)} = 0 \quad \forall \alpha \neq 0.$

- (5) *$P(D)$ has a fundamental solution E with $\text{sing supp } E = \{0\}$.*

- (6) $\lim_{\substack{\xi \in \mathbb{R}^n \\ |\xi| \rightarrow +\infty}} d(\xi) = +\infty.$

- (7) $\lim_{\substack{\zeta \in V \\ |\zeta| \rightarrow +\infty}} |\text{Im } \zeta| = +\infty.$

- (8) *There exist $C > 0$ and a largest $0 < c \leq 1$, with $c \in \mathbb{Q}$, such that for all $\alpha \neq 0$:*

$$\left| \frac{D^\alpha P(\xi)}{P(\xi)} \right| \leq C |\xi|^{-|\alpha|c} \quad \xi \in \mathbb{R}^n, \quad |\xi| \gg 1.$$

- (9) *There exist $C > 0$ and a largest $0 < c \leq 1$, with $c \in \mathbb{Q}$, such that*

$$(2.2) \quad d(\xi) \geq C |\xi|^c, \quad \xi \in \mathbb{R}^n, \quad |\xi| \gg 1.$$

Here $\text{WF}(u)$ and $\text{sing supp } u$ denote the classical wave front set and singular support of $u \in \mathcal{D}'(\Omega)$, as defined in [12].

Remark 2.5. By [13, Lemma 11.1.4] there exists a constant $C > 0$ such that for all polynomials P of degree $\leq m$

$$(2.3) \quad C^{-1} \leq d(\xi) \sum_{\alpha \neq 0} \left| \frac{D^\alpha P(\xi)}{P(\xi)} \right|^{\frac{1}{|\alpha|}} \leq C \quad \text{for } \xi \in \mathbb{R}^n \text{ with } P(\xi) \neq 0.$$

Therefore the constant c at the exponent in (8) and (9) of Theorem 2.4 coincide.

In particular, from (8) with $|\alpha| = m$ we have that if P is hypoelliptic and of order m then there exist $\delta, d > 0$ such that

$$(2.4) \quad |P(\xi)| \geq \delta |\xi|^d \quad \xi \in \mathbb{R}^n, \quad |\xi| \gg 1,$$

with $mc = d \leq m$.

Moreover, by [10, Thm 3.1], there exists a smallest constant $\gamma := \gamma(P) > 0$, which depends on P and will be relevant later such that

$$(2.5) \quad |D^\alpha P(\xi)|^2 \leq C(1 + |P(\xi)|^2)^{1 - \frac{|\alpha|}{\gamma}} \quad \forall \xi \in \mathbb{R}^n \quad \forall \alpha \in \mathbb{N}_0^n,$$

for some $C > 0$. Note that $m \leq \gamma \leq m/c$, since $b := 1/\gamma$ has been determined in Theorem 3.1 of [10] applying the Tarski-Seidenberg theorem to

$$(2.6) \quad M(\lambda) := \sup_{|P(\xi)|=\lambda} |\text{grad } P(\xi)| = A\lambda^{1-b}(1 + o(1)),$$

and (8) of Theorem 2.4 implies

$$(2.7) \quad M(\lambda) \leq A' \lambda^{1-\frac{c}{m}}, \quad \lambda \gg 1$$

for some $A, A' > 0$, if P has order m ; (2.6) and (2.7) imply $b \geq c/m$ and hence $m \leq \gamma \leq m/c$ ($\gamma \geq m$ by (2.5) with $|\alpha| = m$).

We want to generalize Theorem 2.4 to ω -hypoellipticity, for a weight function ω .

Definition 2.6. A linear partial differential operator $P(D)$ with constant coefficients is said to be (ω) -hypoelliptic ($\{\omega\}$ -hypoelliptic) if every solution $u \in \mathcal{D}'(\mathbb{R}^n)$ of $P(D)u = f$ is in $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ whenever $f \in \mathcal{E}_{(\omega)}(\mathbb{R}^n)$ ($u \in \mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$ whenever $f \in \mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$).

We have the following characterization of (ω) -hypoellipticity (Beurling case):

Theorem 2.7. Let $P(D)$ be a linear partial differential operator with constant coefficients and ω a non-quasianalytic weight function. The following conditions are equivalent:

- (1) $P(D)$ is (ω) -hypoelliptic.
- (2) $\lim_{\substack{\zeta \in V \\ |\zeta| \rightarrow +\infty}} \frac{|\operatorname{Im} \zeta|}{\omega(\zeta)} = +\infty$.
- (3) $\lim_{\substack{\xi \in \mathbb{R}^n \\ |\xi| \rightarrow +\infty}} \frac{\omega(\xi)}{d(\xi)} = 0$.
- (4) $\lim_{\substack{\xi \in \mathbb{R}^n \\ |\xi| \rightarrow +\infty}} \frac{\omega(\xi)^{|\alpha|} |D^\alpha P(\xi)|}{|P(\xi)|} = 0 \quad \forall \alpha \neq 0$.

Proof. (1) \Leftrightarrow (2) is proved in [5].

(3) \Rightarrow (2): If $\zeta \in V$ and $|\operatorname{Im} \zeta| \leq |\operatorname{Re} \zeta|$, then from property (α) of ω we have that

$$\frac{|\operatorname{Im} \zeta|}{\omega(\zeta)} \geq \frac{d(\operatorname{Re} \zeta)}{L(\omega(\operatorname{Re} \zeta) + 1)} \rightarrow +\infty$$

by (3).

If $\zeta \in V$ and $|\operatorname{Re} \zeta| \leq |\operatorname{Im} \zeta|$ then

$$\frac{|\operatorname{Im} \zeta|}{\omega(\zeta)} \geq \frac{1}{2} \frac{|\operatorname{Re} \zeta| + |\operatorname{Im} \zeta|}{\omega(\zeta)} \rightarrow +\infty$$

since $\omega(t) = o(t)$.

(2) \Rightarrow (3): For every fixed $\xi \in \mathbb{R}^n$ take $\zeta \in V$ with $|\zeta - \xi| \leq 2d(\xi)$. Take $|\xi|$ large enough.

If $|\zeta - \xi| \leq |\zeta|$, then $|\xi| \leq 2|\zeta|$ and

$$\frac{d(\xi)}{\omega(\xi)} \geq \frac{1}{2} \frac{|\zeta - \xi|}{\omega(\xi)} \geq \frac{1}{2} \frac{|\operatorname{Im} \zeta|}{L(\omega(\zeta) + 1)} \rightarrow +\infty$$

because of (2).

In $|\zeta - \xi| \geq |\zeta|$, then (3) follows from the inequality

$$\frac{d(\xi)}{\omega(\xi)} \geq \frac{1}{2} \max \left\{ \frac{|\xi|}{\omega(\xi)} - \frac{|\zeta|}{\omega(\xi)}, \frac{|\zeta|}{\omega(\xi)} \right\},$$

since $\omega(t) = o(t)$.

(3) \Leftrightarrow (4): follows from (2.3). □

Example 2.8. Theorem 2.7 shows, for example, that the heat operator $P = \partial_t - \Delta_x$ is not $(t^{1/2})$ -hypoelliptic, since

$$\frac{|(\tau, \xi)|^{2a} |D_{x_j}^2 P(\tau, \xi)|}{|P(\tau, \xi)|} = \frac{(\tau^2 + |\xi|^2)^a \cdot 2}{|i\tau + |\xi|^2|} = \frac{2(\tau^2 + |\xi|^2)^a}{\sqrt{\tau^2 + |\xi|^4}} \not\rightarrow 0$$

for $\tau = |\xi| \rightarrow +\infty$ if $2a \geq 1$. On the other hand, it is well known that the heat operator is $\{t^{1/2}\}$ -hypoelliptic (as can be seen also by Theorem 2.9 below).

In the Roumieu case we have the following theorem of characterization of $\{\omega\}$ -hypoellipticity, which generalizes Proposition 2.2.1 of [25] (where a similar result is given for the Gevrey classes):

Theorem 2.9. *Let $P(D)$ be a linear partial differential operator with constant coefficients and ω a non-quasianalytic weight function. The following conditions are equivalent:*

(1) $P(D)$ is $\{\omega\}$ -hypoelliptic.

(2) $\liminf_{\zeta \in V, |\zeta| \rightarrow +\infty} \frac{|\operatorname{Im} \zeta|}{\omega(\zeta)} > 0$.

(3) There exists $c > 0$ such that

$$\omega(\zeta) \leq c(1 + |\operatorname{Im} \zeta|) \quad \forall \zeta \in V.$$

(4) There exist $c, C > 0$ such that

$$\omega(\xi) \leq Cd(\xi) \quad \text{for } \xi \in \mathbb{R}^n, |\xi| > c.$$

(5) There exist $c, C > 0$ such that

$$|D^\alpha P(\xi)| \leq C|P(\xi)|\omega(\xi)^{-|\alpha|} \quad \text{for } \xi \in \mathbb{R}^n, |\xi| > c, \alpha \in \mathbb{N}_0^n.$$

Proof. (1) \Leftrightarrow (2) is proved in [5].

(2) \Leftrightarrow (3): it is easy to check.

(3) \Rightarrow (4): For every fixed $\xi \in \mathbb{R}^n$ take $\zeta \in V$ with $|\zeta - \xi| \leq 2d(\xi)$.

If $|\zeta| \leq |\xi - \zeta|$ then $|\xi| \leq 2|\xi - \zeta|$ and for $|\xi|$ large enough

$$\omega(\xi) \leq \omega(2|\xi - \zeta|) \leq c|\xi - \zeta| \leq 2cd(\xi)$$

for some $c > 0$ since $\omega(t) = o(t)$.

If $|\xi - \zeta| \leq |\zeta|$ then $|\xi| \leq 2|\zeta|$ and, by property (α) of ω and (3),

$$\omega(\xi) \leq L(\omega(\zeta) + 1) \leq L'(|\operatorname{Im} \zeta| + 1) \leq L'(|\zeta - \xi| + 1) \leq L''d(\xi),$$

for some $L', L'' > 0$, since (3) implies that $d(\xi) \rightarrow +\infty$ (see [25, Prop. 2.2.1]).

(4) \Rightarrow (3): If $|\operatorname{Im} \zeta| \leq |\operatorname{Re} \zeta|$, then property (α) of ω and (4) imply that

$$\omega(\zeta) \leq L(\omega(\operatorname{Re} \zeta) + 1) \leq L'(d(\operatorname{Re} \zeta) + 1) \leq L'(|\operatorname{Im} \zeta| + 1)$$

for some $L' > 0$.

If $|\operatorname{Re} \zeta| \leq |\operatorname{Im} \zeta|$ then, by property (α) of ω ,

$$\omega(\zeta) \leq L(\omega(|\operatorname{Im} \zeta|) + 1) \leq L'(|\operatorname{Im} \zeta| + 1)$$

for some $L' > 0$ since $\omega(t) = o(t)$.

(4) \Leftrightarrow (5): It is straightforward because of (2.3). \square

Remark 2.10. From Theorems 2.7 and 2.9 we immediately get the well-known result that (ω) -hypoellipticity implies $\{\omega\}$ -hypoellipticity and they both imply hypoellipticity by Theorem 2.4.

3. Characterization of ω -micro-hypoellipticity with respect to the iterates of an operator

Let ω be a non-quasianalytic weight function and $P(D)$ a linear partial differential operator with constant coefficients. We want to characterize the functions in the class \mathcal{E}_*^P , for $*$ = (ω) or $\{\omega\}$, by means of the Fourier transform. To this aim we use a Paley-Wiener theorem for these spaces of functions, that we borrow from [15]. Since our spaces $\mathcal{E}_*^P(\Omega)$ are slightly different from the analogous ones defined in [15], our Paley-Wiener theorem is only slightly different from the one of [15], and therefore we shall present the suitable statement here, omitting the proof (see [15, Lemma 3.1]).

Lemma 3.1. *Let ω be a non-quasianalytic weight function, P a polynomial of degree m , K a compact convex subset of \mathbb{R}^n and $f \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp} f \subset K$. Then the following statements are equivalent:*

(1) *there exists $\lambda > 0$ such that*

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 e^{\lambda\omega(|P(\xi)|^{1/m})} d\xi < +\infty;$$

(2) *there exists $\lambda, C > 0$ such that*

$$\|P^j(D)f\|_{2,\mathbb{R}^n} \leq C e^{\lambda\varphi^*(\frac{jm}{\lambda})} \quad \forall j \in \mathbb{N}_0.$$

To be more precise, if (1) holds for some $\lambda_0 > 0$, then (2) holds for $\lambda = \lambda_0/2$ and for $C := (2\pi)^{-n/2} \left(\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 e^{\lambda_0\omega(|P(\xi)|^{1/m})} d\xi \right)^{1/2}$.

Vice versa, if (2) holds for some $\lambda > 0$, then

$$(3.1) \quad |\widehat{f}(\zeta)| \leq m(K)^{1/2} C D_{\lambda,\omega} e^{H_K(\operatorname{Im} \zeta) - \frac{\lambda}{2}\omega(|P(\zeta)|^{1/m})} \quad \forall \zeta \in \mathbb{C}^n,$$

for some $D_{\lambda,\omega} > 0$ depending on λ and ω , where $m(K)$ is the Lebesgue measure of K and $H_K(\cdot)$ the supporting function of K ; therefore (1) holds for any $\lambda' < \lambda$.

We present also, similarly as in [15, Thms 3.3 and 3.4], the following Paley-Wiener type theorem for functions in

$$\mathcal{D}_*^P(\mathbb{R}^n) = \{f \in \mathcal{E}_*^P(\mathbb{R}^n) : f \in \mathcal{D}(\mathbb{R}^n)\},$$

for $*$ = (ω) or $\{\omega\}$:

Theorem 3.2. *Let $P(\xi)$ be a hypoelliptic polynomial of degree m and ω a non-quasianalytic weight function. If $f \in \mathcal{D}_{(\omega)}^P(\mathbb{R}^n)$ (resp. $f \in \mathcal{D}_{\{\omega\}}^P(\mathbb{R}^n)$) then its Fourier-Laplace transform $F(\zeta) = \widehat{f}(\zeta)$ is an entire function satisfying the following two conditions:*

(i) *there exist $C, A > 0$ such that*

$$|F(\zeta)| \leq C e^{A|\zeta|} \quad \forall \zeta \in \mathbb{C}^n;$$

(ii) *for every $\lambda > 0$ (resp. there exists $\lambda > 0$):*

$$\int_{\mathbb{R}^n} |F(\xi)|^2 e^{\lambda\omega(|P(\xi)|^{1/m})} d\xi < +\infty.$$

Vice versa, if F is an entire function satisfying (i) and (ii), then $F(\zeta) = \widehat{f}(\zeta)$ for some $f \in \mathcal{D}_{(\omega)}^P(\mathbb{R}^n)$ (resp. $f \in \mathcal{D}_{\{\omega\}}^P(\mathbb{R}^n)$).

Now, we have all the tools to prove the following theorems of characterization of $\mathcal{E}_{(\omega)}^P$ and $\mathcal{E}_{\{\omega\}}^P$:

Theorem 3.3. *Let $P(D)$ be a hypoelliptic linear partial differential operator with constant coefficients, Ω an open subset of \mathbb{R}^n , $u \in \mathcal{D}'(\Omega)$ and $x_0 \in \Omega$. Let ω be a non-quasianalytic weight function such that $\omega(t^\gamma) = o(\sigma(t))$, as $t \rightarrow +\infty$, where γ is the constant defined in (2.5) and $\sigma(t) = t^{1/s}$ for some $s > 1$.*

The following conditions are equivalent:

(1) *$u \in \mathcal{E}_{(\omega)}^P(U)$ for some neighborhood U of x_0 .*

(2) *There exists $\{f_N\}_{N \in \mathbb{N}} \subset \mathcal{E}'(\Omega)$ such that $f_N = P(D)^N u$ in a neighborhood of x_0 and:*

$$(3.2) \quad \begin{aligned} &\forall k \in \mathbb{N}, M \in \mathbb{R} \exists C_{k,M} > 0 : \\ &|\widehat{f_N}(\xi)| \leq C_{k,M} e^{k\varphi^*(Nm/k)} (1 + |\xi|)^M \quad \forall N \in \mathbb{N}, \xi \in \mathbb{R}^n. \end{aligned}$$

(3) *There exists $\psi \in \mathcal{D}_{\{\sigma\}}(\Omega)$, with $\psi \equiv 1$ in a neighborhood of x_0 , such that:*

$$(3.3) \quad \begin{aligned} &\forall k \in \mathbb{N} \exists C_k > 0 : \\ &|\widehat{\psi u}(\xi)| \leq C_k e^{-k\omega(|P(\xi)|^{1/m})} \quad \forall \xi \in \mathbb{R}^n. \end{aligned}$$

(4) There exists $\psi \in \mathcal{D}_{\{\sigma\}}(\Omega)$, with $\psi \equiv 1$ in a neighborhood of x_0 , such that:

$$\forall k \in \mathbb{N}, \ell > 0 \exists C_{k,\ell} > 0 : \quad (3.4) \quad |P(\xi)|^N |\widehat{\psi u}(\xi)| \leq C_{k,\ell} e^{k\varphi^*(Nm/k)} (1 + |\xi|)^{-\ell} \quad \forall N \in \mathbb{N}_0, \xi \in \mathbb{R}^n.$$

Theorem 3.4. Let $P(D)$ be a hypoelliptic linear partial differential operator with constant coefficients, Ω an open subset of \mathbb{R}^n , $u \in \mathcal{D}'(\Omega)$ and $x_0 \in \Omega$. Let ω be a non-quasianalytic weight function such that $\omega(t^\gamma) = o(\sigma(t))$, as $t \rightarrow +\infty$, where γ is the constant defined in (2.5) and $\sigma(t) = t^{1/s}$ for some $s > 1$.

Then the following conditions are equivalent:

{1} $u \in \mathcal{E}_{\{\omega\}}^P(U)$ for some neighborhood U of x_0 .

{2} There exists $\{f_N\}_{N \in \mathbb{N}} \subset \mathcal{E}'(\Omega)$ such that $f_N = P(D)^N u$ in a neighborhood of x_0 and:

$$\exists k \in \mathbb{N}, \forall M \in \mathbb{R} \exists C_M > 0 : \quad (3.5) \quad |\widehat{f_N}(\xi)| \leq C_M e^{\frac{1}{k}\varphi^*(Nm k)} (1 + |\xi|)^M \quad \forall N \in \mathbb{N}, \xi \in \mathbb{R}^n.$$

{3} There exists $\psi \in \mathcal{D}_{\{\sigma\}}(\Omega)$, with $\psi \equiv 1$ in a neighborhood of x_0 , such that:

$$\exists k \in \mathbb{N}, C > 0 : \quad (3.6) \quad |\widehat{\psi u}(\xi)| \leq C e^{-\frac{1}{k}\omega(|P(\xi)|^{1/m})} \quad \forall \xi \in \mathbb{R}^n.$$

{4} There exists $\psi \in \mathcal{D}_{\{\sigma\}}(\Omega)$, with $\psi \equiv 1$ in a neighborhood of x_0 , such that:

$$\exists k \in \mathbb{N}, \forall \ell > 0 \exists C_\ell > 0 : \quad (3.7) \quad |P(\xi)|^N |\widehat{\psi u}(\xi)| \leq C_\ell e^{\frac{1}{k}\varphi^*(Nm k)} (1 + |\xi|)^{-\ell} \quad \forall N \in \mathbb{N}_0, \xi \in \mathbb{R}^n.$$

Before proving Theorems 3.3 and 3.4 we need the following lemma:

Lemma 3.5. Let $\Gamma \subseteq \mathbb{R}^n$ be a cone, $P(D)$ a linear partial differential operator of order m with constant coefficients, Ω an open subset of \mathbb{R}^n , $u \in \mathcal{D}'(\Omega)$, $\psi \in \mathcal{D}(\Omega)$. Then:

(1) Beurling case. The following two conditions are equivalent:

(1.a) for every $k \in \mathbb{N}$ there exists $C_k > 0$ such that

$$|\widehat{\psi u}(\xi)| \leq C_k e^{-k\omega(|P(\xi)|^{1/m})} \quad \forall \xi \in \Gamma;$$

(1.b) for every $k \in \mathbb{N}$, $\ell > 0$ there exists $C_{k,\ell} > 0$ such that

$$|P(\xi)|^N |\widehat{\psi u}(\xi)| \leq C_{k,\ell} e^{k\varphi^*(Nm/k)} (1 + |\xi|)^{-\ell} \quad \forall N \in \mathbb{N}_0, \xi \in \Gamma.$$

(2) Roumieu case. The following two conditions are equivalent:

(2.a) there exist $k \in \mathbb{N}$ and $C > 0$ such that

$$|\widehat{\psi u}(\xi)| \leq C e^{-\frac{1}{k}\omega(|P(\xi)|^{1/m})} \quad \forall \xi \in \Gamma;$$

(2.b) there exists $k \in \mathbb{N}$ such that for every $\ell > 0$ there is $C_\ell > 0$ s.t.

$$|P(\xi)|^N |\widehat{\psi u}(\xi)| \leq C_\ell e^{\frac{1}{k}\varphi^*(Nm k)} (1 + |\xi|)^{-\ell} \quad \forall N \in \mathbb{N}_0, \xi \in \Gamma.$$

Proof of Lemma 3.5. Beurling case. (1.a) \Rightarrow (1.b): Since P is hypoelliptic, by (2.4), (1.a), [3, Lemma 16(i)] and the convexity of φ^* , we have, for all $\ell > 0$ and $N \in \mathbb{N}_0$,

$$\begin{aligned} |\xi|^\ell |P(\xi)|^N |\widehat{\psi u}(\xi)| &\leq \delta^{-\ell/d} |P(\xi)|^{\frac{\ell}{d} + N} |\widehat{\psi u}(\xi)| \\ &\leq \delta^{-\ell/d} (|P(\xi)|^{\frac{1}{m}})^{\frac{\ell m}{d} + mN} C_{2k} e^{-2k\omega(|P(\xi)|^{\frac{1}{m}})} \\ &\leq C_{k,\ell} e^{k\varphi^*(\frac{Nm}{k})}. \end{aligned}$$

(1.b) \Rightarrow (1.a): Assume first that $|P(\xi)| > 1$. By (1.b) with $\ell = m$ we have that for every k there exists a constant $C_{k,m} = C_k > 0$ such that

$$|\xi|^m |P(\xi)|^N e^{-k\varphi^*(\frac{Nm}{k})} |\widehat{\psi u}(\xi)| \leq C_k \quad \forall N \in \mathbb{N}, \xi \in \Gamma.$$

Since $|P(\xi)| \leq c|\xi|^m$ for some $c > 0$, we thus have that

$$(3.8) \quad \sup_{N \in \mathbb{N}_0} \left\{ (|P(\xi)|^{1/m})^{(N+1)m} e^{-k\varphi^*(\frac{Nm}{k})} \right\} |\widehat{\psi u}(\xi)| \leq C'_k$$

for some $C'_k > 0$. But for all $s > 0$ there exists $N \in \mathbb{N}_0$ such that $Nm \leq s < (N+1)m$, so that

$$\begin{aligned} \sup_{N \in \mathbb{N}_0} \left\{ (|P(\xi)|^{1/m})^{(N+1)m} e^{-k\varphi^*(\frac{Nm}{k})} \right\} &\geq \sup_{s > 0} \left\{ (|P(\xi)|^{1/m})^s e^{-k\varphi^*(\frac{s}{k})} \right\} \\ &= \exp\{k\varphi(\log |P(\xi)|^{1/m})\} = e^{k\omega(|P(\xi)|^{1/m})}. \end{aligned}$$

Substituting in (3.8) we have that

$$e^{k\omega(|P(\xi)|^{1/m})} |\widehat{\psi u}(\xi)| \leq C'_k \quad \forall \xi \in \Gamma \text{ with } |P(\xi)| > 1.$$

If $|P(\xi)| \leq 1$ then $\omega(|P(\xi)|^{1/m}) \equiv 0$ and the thesis is trivial.

Roumieu case. It is similar to the Beurling case. □

Proof of Theorem 3.3. (1) \Leftrightarrow (2) was proved in [3, Prop. 6].

(1) \Rightarrow (3): Let $u \in \mathcal{E}_{(\omega)}^P(U)$ and take $\psi \in \mathcal{D}_{\{\sigma\}}(U)$ with $\psi \equiv 1$ in a neighborhood $V \subset U$ of x_0 . Since $\omega(t^\gamma) = o(\sigma(t))$, by Proposition 17 of [3] we have that $\psi u \in \mathcal{D}_{(\omega)}^P(U)$.

By the Paley-Wiener Theorem 3.2, for all $\lambda > 0$ there exists $C_\lambda > 0$ such that

$$\int_{\mathbb{R}^n} |\widehat{\psi u}(\xi)|^2 e^{4\lambda\omega(|P(\xi)|^{1/m})} d\xi \leq C_\lambda$$

and hence, by Lemma 3.1 for $K = \text{supp } \psi$,

$$|\widehat{\psi u}(\zeta)| \leq m(K)^{1/2} C_\lambda D_{\lambda,\omega} e^{H_K(\text{Im } \zeta) - \lambda\omega(|P(\zeta)|^{1/m})} \quad \forall \zeta \in \mathbb{C}^n.$$

For $\lambda = k \in \mathbb{N}$ and $\zeta = \xi \in \mathbb{R}^n$ we thus obtain (3.3).

(3) \Rightarrow (1): From (3.3) it follows that for every $\lambda > 0$

$$C_\lambda := \left(\int_{\mathbb{R}^n} |\widehat{\psi u}(\xi)|^2 e^{2\lambda\omega(|P(\xi)|^{1/m})} d\xi \right)^{1/2} < +\infty.$$

Therefore, by Lemma 3.1, we obtain

$$\|P^j(D)(\psi u)\|_{2,\mathbb{R}^n} \leq (2\pi)^{-n/2} C_\lambda e^{\lambda\varphi^*(\frac{jm}{\lambda})} \quad \forall j \in \mathbb{N}_0.$$

Since $\psi \equiv 1$ in a neighborhood U of x_0 , then for every compact $K \subset\subset U$

$$\|P^j(D)u\|_{2,K} = \|P^j(D)(\psi u)\|_{2,K} \leq \|P^j(D)(\psi u)\|_{2,\mathbb{R}^n} \leq C'_\lambda e^{\lambda\varphi^*(\frac{jm}{\lambda})} \quad \forall j \in \mathbb{N}_0,$$

for some $C'_\lambda > 0$, i.e. $u \in \mathcal{E}_{(\omega)}^P(U)$.

(3) \Leftrightarrow (4) follows from Lemma 3.5 for $\Gamma = \mathbb{R}^n$. □

Proof of Theorem 3.4. It is analogous to that of Theorem 3.3 for $\lambda = 1/k$. □

Let us now consider the wave front set with respect to the iterates of an operator. We recall the following definition from [3]:

Definition 3.6. Let $P(D)$ be a hypoelliptic linear partial differential operator with constant coefficients, ω a non-quasianalytic weight function, Ω an open subset of \mathbb{R}^n , $u \in \mathcal{D}'(\Omega)$. We say that a point $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ is not in the (ω) -wave front set $\text{WF}_{(\omega)}^P(u)$ (resp. $\{\omega\}$ -wave front set $\text{WF}_{\{\omega\}}^P(u)$) with respect to the iterates of P , if there are a neighborhood U of x_0 , an open conic neighborhood Γ of ξ_0 and a sequence $\{f_N\}_{N \in \mathbb{N}} \subset \mathcal{E}'(\Omega)$ such that (i) and (ii) (resp. (i) and (iii)) of the following conditions hold:

(i) $f_N = P(D)^N u$ in U .

(ii) Beurling case:

(a) there exist $M, C > 0$ such that for all $k \in \mathbb{N}$ there is $C_k > 0$:

$$(3.9) \quad |\widehat{f_N}(\xi)| \leq C_k C^N (e^{\frac{k}{Nm}\varphi^*(\frac{Nm}{k})} + |\xi|)^{Nm} (1 + |\xi|)^M \quad \forall N \in \mathbb{N}, \xi \in \mathbb{R}^n;$$

(b) for every $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$ there exists $C_{k,\ell} > 0$ such that

$$(3.10) \quad |\widehat{f}_N(\xi)| \leq C_{k,\ell} e^{k\varphi^*(Nm/k)} (1 + |\xi|)^{-\ell} \quad \forall N \in \mathbb{N}, \xi \in \Gamma.$$

(iii) Roumieu case:

(a) there exist $k \in \mathbb{N}$, $M, C > 0$ such that

$$(3.11) \quad |\widehat{f}_N(\xi)| \leq C^N (e^{\frac{1}{Nm} \varphi^*(Nm)} + |\xi|)^{Nm} (1 + |\xi|)^M \quad \forall N \in \mathbb{N}, \xi \in \mathbb{R}^n;$$

(b) there exists $k \in \mathbb{N}$ such that for every $\ell \in \mathbb{N}_0$ there is $C_\ell > 0$ s.t.

$$(3.12) \quad |\widehat{f}_N(\xi)| \leq C_\ell e^{\frac{1}{k} \varphi^*(Nm)} (1 + |\xi|)^{-\ell} \quad \forall N \in \mathbb{N}, \xi \in \Gamma.$$

Now we want to characterize the ω -wave front set in terms of condition (3.3) for the Beurling case, and (3.6) for the Roumieu case. To do this, we first give the following definition of wave front set and we prove in Theorem 3.13 below its equivalence to the one of Definition 3.6:

Definition 3.7. Let $P(D)$ be a hypoelliptic linear partial differential operator with constant coefficients; let ω be a non-quasianalytic weight function with $\omega(t^\gamma) = o(\sigma(t))$, as $t \rightarrow +\infty$, where γ is the constant defined in (2.5) and $\sigma(t) = t^{1/s}$ for some $s > 1$; let Ω be an open subset of \mathbb{R}^n and $u \in \mathcal{D}'(\Omega)$.

We say that $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ is not in the (ω) -wave front set $\text{WF}_{(\omega),P}(u)$ (resp. $\{\omega\}$ -wave front set $\text{WF}_{\{\omega\},P}(u)$) with respect to the iterates of P , if there exist a neighborhood U of x_0 , an open conic neighborhood Γ of ξ_0 and $\psi \in \mathcal{D}_{\{\sigma\}}(\Omega)$ with $\psi \equiv 1$ in U such that the following condition (i) (resp. (ii)) holds:

(i) Beurling case: For every $k \in \mathbb{N}$ there exists $C_k > 0$ such that

$$(3.13) \quad |\widehat{\psi u}(\xi)| \leq C_k e^{-k\omega(|P(\xi)|^{1/m})} \quad \forall \xi \in \Gamma.$$

(ii) Roumieu case: There exist $k \in \mathbb{N}$, $C > 0$ such that

$$(3.14) \quad |\widehat{\psi u}(\xi)| \leq C e^{-\frac{1}{k} \omega(|P(\xi)|^{1/m})} \quad \forall \xi \in \Gamma.$$

In order to prove that $\text{WF}_*^P(u) = \text{WF}_{*,P}(u)$, for $*$ = (ω) or $\{\omega\}$, let's start by the following:

Lemma 3.8. Let $P(D)$ be a hypoelliptic linear partial differential operator of order m with constant coefficients; let ω be a non-quasianalytic weight function such that $\omega(t^\gamma) = o(\sigma(t))$, as $t \rightarrow +\infty$, where γ is the constant defined in (2.5) and $\sigma(t) = t^{1/s}$ for some $s > 1$; let Ω be an open subset of \mathbb{R}^n and $u \in \mathcal{D}'(\Omega)$. Then

$$\text{WF}_*^P(u) \subseteq \text{WF}_{*,P}(u),$$

for $*$ = (ω) or $\{\omega\}$,

Proof. Beurling case. Let $(x_0, \xi_0) \notin \text{WF}_{(\omega),P}(u)$. There exist then a neighborhood U of x_0 , an open conic neighborhood Γ of ξ_0 and $\psi \in \mathcal{D}_{\{\sigma\}}(\Omega)$ with $\psi \equiv 1$ in U satisfying (3.13). Setting $f_N = P(D)^N(\psi u)$ we have that $f_N \in \mathcal{E}'(\Omega)$, $f_N = P(D)^N u$ in U and

$$(3.15) \quad |\widehat{f}_N(\xi)| \leq |P(\xi)|^N |\widehat{\psi u}(\xi)| \leq C^N (1 + |\xi|)^{Nm} (1 + |\xi|)^M \quad \forall \xi \in \mathbb{R}^n$$

for some $M > 0$, since $\psi u \in \mathcal{E}'(\mathbb{R}^n)$. Clearly (3.15) implies (3.9).

To prove (3.10) take $k \in \mathbb{N}$, $\ell > 0$ and $\xi \in \Gamma$. By (2.4) and (3.13) we have

$$\begin{aligned} |\xi|^\ell |\widehat{f}_N(\xi)| &\leq \delta^{-\ell/d} |P(\xi)|^{\ell/d} |P(\xi)|^N |\widehat{\psi u}(\xi)| \\ &\leq \delta^{-\ell/d} C_{2k} (|P(\xi)|^{1/m})^{\frac{\ell m}{d} + Nm} e^{-2k\omega(|P(\xi)|^{1/m})} \\ &\leq C_{k,\ell} e^{k\varphi^*(\frac{Nm}{k})}, \end{aligned}$$

by [3, Lemma 16(i)] and the convexity of φ^* .

This proves (3.10) and hence $(x_0, \xi_0) \notin \text{WF}_{(\omega)}^P(u)$.

Roumieu case. It's similar to the Beurling case. □

We recall, from [9, Lemma 4] (see also [25, Proposition 3.4.4]), the following lemma that we shall need later:

Lemma 3.9. *Let Γ and Γ' be two cones in \mathbb{R}^n such that $\Gamma' \subset\subset \Gamma$ in the sense that $\Gamma' \cap S^{n-1} \subset\subset \Gamma \cap S^{n-1}$, where S^{n-1} is the unit sphere in \mathbb{R}^n .*

Then there exists a bounded $\phi \in \mathcal{E}_{(\omega)}(\mathbb{R}^n) \subset \mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$ with $\text{supp } \phi \subset \Gamma$, $\phi \equiv 1$ on Γ' (for large $|\xi|$), which is the symbol of a pseudo-differential operator $\phi(D)$ satisfying

$$(3.16) \quad \widehat{\phi(D)u}(\xi) = \phi(\xi)\widehat{u}(\xi) \quad u \in \mathcal{D}_{(\omega)}(\mathbb{R}^n), \quad \xi \in \mathbb{R}^n.$$

Remark 3.10. Here, the definition of pseudodifferential operator is as in [9, Def. 3]. Then, we must consider the symbol of the operator $\phi(D)$ as $(2\pi)^{-n}\phi(\xi)$ (compare with the beginning of the proof of [9, Theorem 2]).

We can now prove the following result:

Theorem 3.11. *Let $P(D)$ be a hypoelliptic linear partial differential operator of order m with constant coefficients; let ω be a non-quasianalytic weight function such that $\omega(t^\gamma) = o(\sigma(t))$, as $t \rightarrow +\infty$, where γ is the constant defined in (2.5) and $\sigma(t) = t^{1/s}$ for some $s > 1$. Let Ω be an open subset of \mathbb{R}^n , $u \in \mathcal{E}'(\Omega)$ and $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$.*

Then $(x_0, \xi_0) \notin \text{WF}_^P(u)$ if and only if there exists ϕ as in Lemma 3.9 such that $\phi(D)u|_V \in \mathcal{E}_*^P(V)$ for some neighborhood V of x_0 , where $*$ = (ω) or $\{\omega\}$.*

Proof. Beurling case. Let $(x_0, \xi_0) \notin \text{WF}_{(\omega)}^P(u)$. There exist a neighborhood U of x_0 , an open conic neighborhood Γ of ξ_0 and a sequence $\{f_N\}_{N \in \mathbb{N}} \subset \mathcal{E}'(\Omega)$ with $f_N = P(D)^N u$ in U and satisfying (3.9) and (3.10).

Now, we consider a conic neighborhood Γ' of ξ_0 with $\Gamma' \subset\subset \Gamma$. Take then ϕ as in Lemma 3.9 and define $h_N := \phi(D)f_N$. Since $\phi(D)$ and $P(D)^N$ commute, we have that $h_N = P(D)^N \phi(D)u$ in a neighborhood of x_0 and $\widehat{h}_N(\xi) = \phi(\xi)\widehat{f}_N(\xi)$ satisfies (3.2) because of (3.10) and ϕ bounded with $\text{supp } \phi \subset \Gamma$. Therefore $\phi(D)u|_V \in \mathcal{E}_{(\omega)}^P(V)$ for a neighborhood V of x_0 by Theorem 3.3.

Vice versa, let us now assume $\phi(D)u|_V \in \mathcal{E}_{(\omega)}^P(V)$, where ϕ is as in Lemma 3.9, for some neighborhood V of x_0 and for some neighborhoods $\Gamma' \subset\subset \Gamma$ of ξ_0 . Take $\varphi \in \mathcal{D}_{\{\sigma\}}(V)$ with $\varphi \equiv 1$ in a neighborhood of x_0 and write

$$\varphi u = \varphi \phi(D)u + \varphi(I - \phi(D))u =: u_1 + u_2.$$

Since $\phi(D)u|_V \in \mathcal{E}_{(\omega)}^P(V)$ by assumption and $\varphi \in \mathcal{D}_{\{\sigma\}}(V)$, we can apply [3, Prop. 17] to get $u_1 \in \mathcal{E}_{(\omega)}^P(V)$. We thus have to prove that $(x_0, \xi_0) \notin \text{WF}_{(\omega)}^P(u_2)$. By Lemma 3.8 it's enough to prove that $(x_0, \xi_0) \notin \text{WF}_{(\omega), P}(u_2)$. To do this, we have to find $\psi \in \mathcal{D}_{\{\sigma\}}(\Omega)$ with $\psi \equiv 1$ in a neighborhood of x_0 and $\widehat{\psi u_2}(\xi)$ satisfying (3.13) in a conic neighborhood of ξ_0 .

Taking $\psi \in \mathcal{D}_{\{\sigma\}}(\Omega)$ with $\psi \equiv 1$ on $\text{supp } \varphi$ we have that $\psi u_2 = u_2$ by the definition of u_2 . Hence we will prove (3.13) for \widehat{u}_2 . We have

$$(3.17) \quad \begin{aligned} \widehat{u}_2 &= \mathcal{F}(\varphi(I - \phi(D))u)(\xi) = (2\pi)^{-n}(\widehat{\varphi} * (1 - \phi)\widehat{u})(\xi) \\ &= (2\pi)^{-n} \int \widehat{\varphi}(\xi - \eta)(1 - \phi)(\eta)\widehat{u}(\eta)d\eta. \end{aligned}$$

Let Λ be a conic neighborhood of ξ_0 with $\bar{\Lambda} \subset \Gamma'$, so that $(1 - \phi)(\eta) = 0$ in Λ . For fixed $\xi \in \Lambda$ we set

$$\begin{aligned} A &:= \{\eta \in \mathbb{R}^n : |\xi - \eta| \leq \delta(|\xi| + |\eta|)\} \\ B &:= \{\eta \in \mathbb{R}^n : |\xi - \eta| > \delta(|\xi| + |\eta|)\} \end{aligned}$$

for $\delta > 0$ small enough so that $A \subset \Gamma'$ and hence $(1 - \phi)(\eta) = 0$ in A .

Splitting the integral (3.17) we have, for $\xi \in \Lambda$:

$$(3.18) \quad \begin{aligned} \widehat{u}_2(\xi) &= (2\pi)^{-n} \int_A \widehat{\varphi}(\xi - \eta)(1 - \phi)(\eta)\widehat{u}(\eta)d\eta + (2\pi)^{-n} \int_B \widehat{\varphi}(\xi - \eta)(1 - \phi)(\eta)\widehat{u}(\eta)d\eta \\ &= (2\pi)^{-n} \int_B \widehat{\varphi}(\xi - \eta)(1 - \phi)(\eta)\widehat{u}(\eta)d\eta. \end{aligned}$$

For $\xi \in \Lambda$ and $\eta \in B$, as $\varphi \in \mathcal{D}_{\{\sigma\}}(\Omega)$ and $\sigma(t) = t^{1/s}$, there exist $C, C', \varepsilon, \varepsilon' > 0$ such that

$$|\widehat{\varphi}(\xi - \eta)| \leq C e^{-\varepsilon \sigma(|\xi - \eta|)} \leq C' e^{-\varepsilon'(\sigma(|\xi|) + \sigma(|\eta|))}.$$

Moreover, $|\widehat{u}(\eta)| \leq C(1 + |\eta|)^M$ for some $C, M > 0$ since $u \in \mathcal{E}'(\Omega)$, and therefore

$$\begin{aligned} \left| \int_B \widehat{\varphi}(\xi - \eta)(1 - \phi)(\eta) \widehat{u}(\eta) d\eta \right| &\leq C' e^{-\varepsilon'(\sigma(\xi))} \int |1 - \phi(\eta)| |\widehat{u}(\eta)| e^{-\varepsilon'(\sigma(\eta))} d\eta \\ &\leq D e^{-\varepsilon'(\sigma(\xi))} \int (1 + |\eta|)^M e^{-\varepsilon'(\sigma(\eta))} d\eta \\ (3.19) \quad &\leq D' e^{-\varepsilon'(\sigma(\xi))}, \end{aligned}$$

for some $D, D' > 0$, because $1 - \phi(\eta)$ is bounded.

Since $\omega(t^\gamma) = o(\sigma(t))$ with $\gamma \geq 1$ (see Remark 2.5), for every $k \in \mathbb{N}$ there exists $R_k > 0$ such that for all $|\xi| \geq R_k$:

$$(3.20) \quad \omega(|P(\xi)|^{\frac{1}{m}}) \leq \omega(c|\xi|) \leq L(\omega(\xi) + 1) \leq \frac{1}{k}\sigma(\xi) + L$$

for some $c, L > 0$, and hence, from (3.19) and (3.18), for every $k \in \mathbb{N}$ there is $D_k > 0$ such that:

$$|\widehat{u}_2(\xi)| \leq D_k e^{-k\omega(|P(\xi)|^{1/m})} \quad \forall \xi \in \Lambda,$$

proving that $(x_0, \xi_0) \notin \text{WF}_{(\omega), P}(u_2)$ and hence $(x_0, \xi_0) \notin \text{WF}_{(\omega)}^P(u)$.

Roumieu case. The proof is analogous to that of the Beurling case. If $(x_0, \xi_0) \notin \text{WF}_{\{\omega\}}^P(u)$ we find a neighborhood U of x_0 , an open conic neighborhood Γ of ξ_0 and a sequence $\{f_N\}_{N \in \mathbb{N}} \subset \mathcal{E}'(\Omega)$ satisfying (i), (iii)(a) and (iii)(b) of Definition 3.6. As in the Beurling case $h_N := \phi(D)f_N$ satisfies the desired estimate (3.5), so that $\phi(D)u|_V \in \mathcal{E}_{\{\omega\}}^P(V)$ for a neighborhood V of x_0 by Theorem 3.4.

Conversely, if $\phi(D)u|_V \in \mathcal{E}_{\{\omega\}}^P(V)$ we proceed as in the Beurling case and obtain that $u_1 := \varphi\phi(D)u \in \mathcal{E}_{\{\omega\}}^P(V)$ by Proposition 17 of [3]. The proof that $(x_0, \xi_0) \notin \text{WF}_{\{\omega\}}^P(u_2)$ is analogous to the one in the Beurling case, applying (3.20) for $k = 1$. \square

Proposition 3.12. *Let $P(D)$ be a hypoelliptic linear partial differential operator with constant coefficients; let ω be a non-quasianalytic weight function such that $\omega(t^\gamma) = o(\sigma(t))$, as $t \rightarrow +\infty$, where γ is the constant defined in (2.5) and $\sigma(t) = t^{1/s}$ for some $s > 1$; let Ω be an open subset of \mathbb{R}^n and $u \in \mathcal{D}'(\Omega)$. If $\psi \in \mathcal{D}_{\{\sigma\}}(\Omega)$, then*

$$\text{WF}_*^P(\psi u) \subseteq \text{WF}_*^P(u),$$

for $*$ = (ω) or $\{\omega\}$.

Proof. **Beurling case.** Let $(x_0, \xi_0) \notin \text{WF}_{(\omega)}^P(u)$. First, we observe that we can assume $u \in \mathcal{E}'(\Omega)$, since the definition of the wave front set is local. Then, there exist a neighborhood V of x_0 and a conic neighborhood Γ of ξ_0 such that

$$(V \times \Gamma) \cap \text{WF}_{(\omega)}^P(u) = \emptyset.$$

For $\Gamma' \subset \subset \Gamma$ there exists, by Theorem 3.11, a bounded $\phi \in \mathcal{E}_{(\omega)}(\mathbb{R}^n)$ with $\text{supp } \phi \subset \Gamma$, $\phi \equiv 1$ in Γ' such that $\phi(D)u|_{V'} \in \mathcal{E}_{(\omega)}^P(V')$ for some neighborhood $V' \subseteq V$ of x_0 .

Let us then consider $\varphi \in \mathcal{D}_{\{\sigma\}}(V')$ with $\varphi \equiv 1$ in a neighborhood of x_0 , and set

$$\varphi\psi u = \varphi\psi\phi(D)u + \varphi\psi(I - \phi(D))u =: u_1 + u_2.$$

Since $\varphi, \psi \in \mathcal{D}_{\{\sigma\}}(\Omega)$ we have, by [7, Prop. 4.4], that $\varphi\psi \in \mathcal{D}_{\{\sigma\}}(\Omega)$ and hence $u_1 \in \mathcal{E}_{(\omega)}^P(V')$ by [3, Prop. 17] because $\phi(D)u|_{V'} \in \mathcal{E}_{(\omega)}^P(V')$.

Arguing as in the proof of Theorem 3.11 with $\varphi\psi \in \mathcal{D}_{\{\sigma\}}$ instead of $\varphi \in \mathcal{D}_{\{\sigma\}}$, we obtain that $(x_0, \xi_0) \notin \text{WF}_{(\omega), P}(u_2)$ and hence $(x_0, \xi_0) \notin \text{WF}_{(\omega)}^P(u_2)$ by Lemma 3.8.

Therefore $(x_0, \xi_0) \notin \text{WF}_{(\omega)}^P(\psi u)$.

Roumieu case. It is similar to the Beurling case. \square

Theorem 3.13. *Let $P(D)$ be a hypoelliptic linear partial differential operator of order m with constant coefficients; let ω be a non-quasianalytic weight function such that $\omega(t^\gamma) = o(\sigma(t))$, as $t \rightarrow +\infty$, where γ is the constant defined in (2.5) and $\sigma(t) = t^{1/s}$ for some $s > 1$; let Ω be an open subset of \mathbb{R}^n and $u \in \mathcal{D}'(\Omega)$. Then*

$$\text{WF}_*^P(u) = \text{WF}_{*,P}(u),$$

for $*$ = (ω) or $\{\omega\}$,

Proof. The inclusion

$$\text{WF}_*^P(u) \subseteq \text{WF}_{*,P}(u)$$

has been proved in Lemma 3.8.

Let us prove the other inclusion.

Beurling case. Let $(x_0, \xi_0) \notin \text{WF}_{(\omega)}^P(u)$. There exist a compact neighborhood K of x_0 and a closed conic neighborhood F of ξ_0 such that

$$\text{WF}_{(\omega)}^P(u) \cap (K \times F) = \emptyset.$$

Take, according to [11, Lemma 2.2], $\chi_N \in \mathcal{D}(K)$ with $\chi_N \equiv 1$ in a neighborhood $K' \subset K$ of x_0 , that satisfies

$$(3.21) \quad \sup_K |D^{\alpha+\beta} \chi_N| \leq C_\alpha (C_\alpha N)^{|\beta|} \quad \forall \alpha, \beta \in \mathbb{N}_0, |\beta| \leq N.$$

Fix $\psi \in \mathcal{D}_{\{\sigma\}}(K')$. By Proposition 3.12

$$\text{WF}_{(\omega)}^P(\psi u) \subseteq \text{WF}_{(\omega)}^P(u),$$

and, hence,

$$\text{WF}_{(\omega)}^P(\psi u) \cap (K \times F) = \emptyset.$$

Now, consider $g_N := \chi_{Nmp} P(D)^N \psi u$, for p sufficiently large so that, by [3, Corollary 10], for every $k, \ell \in \mathbb{N}$ there exists $C_{k,\ell} > 0$ such that

$$(3.22) \quad |\widehat{g}_N(\xi)| \leq C_{k,\ell} e^{k\varphi^*(\frac{Nm}{k})} (1 + |\xi|)^{-\ell} \quad \forall N \in \mathbb{N}, \xi \in F.$$

Moreover,

$$g_N = \chi_{Nmp} P(D)^N \psi u = P(D)^N \psi u \quad \text{in } \mathbb{R}^n,$$

since $\chi_{Nmp} \equiv 1$ on $\text{supp } \psi$.

Therefore

$$|P(\xi)^N \widehat{\psi u}(\xi)| = |\mathcal{F}(P(D)^N \psi u)(\xi)| = |\widehat{g}_N(\xi)| \leq C_{k,\ell} e^{k\varphi^*(\frac{Nm}{k})} (1 + |\xi|)^{-\ell} \quad \forall N \in \mathbb{N}, \xi \in F.$$

By Lemma 3.5, this implies that

$$|\widehat{\psi u}(\xi)| \leq C_k e^{-k\omega(|P(\xi)|^{1/m})} \quad \forall \xi \in F,$$

i.e. $(x_0, \xi_0) \notin \text{WF}_{(\omega),P}(u)$.

Roumieu case. It is similar to the Beurling case. □

By Theorem 3.13 and Lemma 3.5 we can now prove the following characterizations of the wave front set with respect to the iterates:

Corollary 3.14. *Let $P(D)$ be a hypoelliptic linear partial differential operator with constant coefficients; let ω be a non-quasianalytic weight function such that $\omega(t^\gamma) = o(\sigma(t))$, as $t \rightarrow +\infty$, where γ is the constant defined in (2.5) and $\sigma(t) = t^{1/s}$ for some $s > 1$; let Ω be an open subset of \mathbb{R}^n , $u \in \mathcal{D}'(\Omega)$ and $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$.*

The following conditions are equivalent:

$$(1) \quad (x_0, \xi_0) \notin \text{WF}_{(\omega)}^P(u).$$

- (2) There exist a neighborhood U of x_0 , an open conic neighborhood Γ of ξ_0 and $\psi \in \mathcal{D}_{\{\sigma\}}(\Omega)$ with $\psi \equiv 1$ in U such that:

$$(3.23) \quad \begin{aligned} & \forall k \in \mathbb{N}, \exists C_k > 0 : \\ & |\widehat{\psi u}(\xi)| \leq C_k e^{-k\omega(|P(\xi)|^{1/m})} \quad \forall \xi \in \Gamma. \end{aligned}$$

- (3) There exist a neighborhood U of x_0 , an open conic neighborhood Γ of ξ_0 and $\psi \in \mathcal{D}_{\{\sigma\}}(\Omega)$ with $\psi \equiv 1$ in U such that:

$$(3.24) \quad \begin{aligned} & \forall k, \ell \in \mathbb{N}, \exists C_{k,\ell} > 0 : \\ & |P(\xi)|^N |\widehat{\psi u}(\xi)| \leq C_{k,\ell} e^{k\varphi^*(\frac{Nm}{k})} (1 + |\xi|)^{-\ell} \quad \forall N \in \mathbb{N}, \xi \in \Gamma. \end{aligned}$$

- (4) There exist a neighborhood U of x_0 , an open conic neighborhood Γ of ξ_0 and a bounded sequence $\{u_N\}_{N \in \mathbb{N}} \subset \mathcal{E}'(\Omega)$ such that $u_N = u$ in U and:

$$(3.25) \quad \begin{aligned} & \forall k \in \mathbb{N}, \ell \in \mathbb{N}_0, \exists C_{k,\ell} > 0 : \\ & |P(\xi)|^N |\widehat{u}_N(\xi)| \leq C_{k,\ell} e^{k\varphi^*(\frac{Nm}{k})} (1 + |\xi|)^{-\ell} \quad \forall N \in \mathbb{N}, \xi \in \Gamma. \end{aligned}$$

Corollary 3.15. Let $P(D)$ be a hypoelliptic linear partial differential operator with constant coefficients; let ω be a non-quasianalytic weight function such that $\omega(t^\gamma) = o(\sigma(t))$, as $t \rightarrow +\infty$, where γ is the constant defined in (2.5) and $\sigma(t) = t^{1/s}$ for some $s > 1$; let Ω be an open subset of \mathbb{R}^n , $u \in \mathcal{D}'(\Omega)$ and $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$.

The following conditions are equivalent:

- (1) $(x_0, \xi_0) \notin \text{WF}_{\{\omega\}}^P(u)$.
- (2) There exist a neighborhood U of x_0 , an open conic neighborhood Γ of ξ_0 and $\psi \in \mathcal{D}_{\{\sigma\}}(\Omega)$ with $\psi \equiv 1$ in U such that:

$$(3.26) \quad \begin{aligned} & \exists k \in \mathbb{N}, C > 0 : \\ & |\widehat{\psi u}(\xi)| \leq C e^{-\frac{1}{k}\omega(|P(\xi)|^{1/m})} \quad \forall \xi \in \Gamma. \end{aligned}$$

- (3) There exist a neighborhood U of x_0 , an open conic neighborhood Γ of ξ_0 and $\psi \in \mathcal{D}_{\{\sigma\}}(\Omega)$ with $\psi \equiv 1$ in U such that:

$$(3.27) \quad \begin{aligned} & \exists k \in \mathbb{N}, \forall \ell \in \mathbb{N}, \exists C_\ell > 0 : \\ & |P(\xi)|^N |\widehat{\psi u}(\xi)| \leq C_\ell e^{\frac{1}{k}\varphi^*(Nm k)} (1 + |\xi|)^{-\ell} \quad \forall N \in \mathbb{N}, \xi \in \Gamma. \end{aligned}$$

- (4) There exist a neighborhood U of x_0 , an open conic neighborhood Γ of ξ_0 and a bounded sequence $\{u_N\}_{N \in \mathbb{N}} \subset \mathcal{E}'(\Omega)$ such that $u_N = u$ in U and:

$$(3.28) \quad \begin{aligned} & \exists k \in \mathbb{N}, \forall \ell \in \mathbb{N}_0, \exists C_\ell > 0 : \\ & |P(\xi)|^N |\widehat{u}_N(\xi)| \leq C_\ell e^{\frac{1}{k}\varphi^*(Nm k)} (1 + |\xi|)^{-\ell} \quad \forall N \in \mathbb{N}, \xi \in \Gamma. \end{aligned}$$

Proof of Corollary 3.14. (1) \Leftrightarrow (2) follows from Theorem 3.13.

(2) \Leftrightarrow (3) follows from Lemma 3.5.

(3) \Rightarrow (4): Taking $u_N = \psi u$ we have that $\{u_N\}_{N \in \mathbb{N}}$ is a bounded sequence in $\mathcal{E}'(\Omega)$, $u_N = u$ in U and (3.24) implies (3.25) by the choice of u_N .

(4) \Rightarrow (1): Taking $f_N = P(D)^N u_N$ we have that $f_N = P(D)^N u$ in U and, by (3.25),

$$|\widehat{f}_N(\xi)| = |P(\xi)|^N |\widehat{u}_N(\xi)| \leq C_{k,\ell} e^{k\varphi^*(\frac{Nm}{k})} (1 + |\xi|)^{-\ell} \quad \forall \xi \in \Gamma,$$

giving condition (3.10).

Finally, since $\{u_N\}_{N \in \mathbb{N}}$ is a bounded sequence in $\mathcal{E}'(\Omega)$, there exist $c, M > 0$ such that

$$|\widehat{u}_N(\xi)| \leq c(1 + |\xi|)^M \quad \forall \xi \in \mathbb{R}^n$$

and hence

$$\begin{aligned} |\widehat{f}_N(\xi)| & \leq |P(\xi)|^N |\widehat{u}_N(\xi)| \leq C^N (1 + |\xi|)^{mN} c(1 + |\xi|)^M \\ & \leq \tilde{C}^N (e^{\frac{k}{Nm}\varphi^*(\frac{Nm}{k})} + |\xi|)^{Nm} (1 + |\xi|)^M \quad \forall \xi \in \mathbb{R}^n, \end{aligned}$$

for some $C, \tilde{C} > 0$, proving also (3.9).

Therefore $(x_0, \xi_0) \notin \text{WF}_{(\omega)}^P(u)$. □

Proof of Corollary 3.15. Its similar to the Beurling case, Corollary 3.14. □

The new characterization of $\text{WF}_*^P(u)$ given by Corollaries 3.14 and 3.15 allows to complete Theorem 18 of [3], obtaining the existence of a distribution with prescribed ω -wave front set with respect to the iterates:

Theorem 3.16. *Let $P(D)$ be a linear partial differential operator of order m with constant coefficients which is hypoelliptic, but not elliptic. Let ω be a non-quasi-analytic weight function such that $\omega(t^b) = o(\bar{\sigma}(t))$, as $t \rightarrow +\infty$, where $\bar{\sigma}(t) = t^{1/s}$ for some $s > 1$ and $b = \max\{\gamma, 3/2\}$, with γ defined in (2.5).*

Given an open subset Ω of \mathbb{R}^n and a closed conic subset S of $\Omega \times (\mathbb{R}^n \setminus \{0\})$, there exists $u \in \mathcal{D}'(\Omega)$ with

$$\text{WF}_*^P(u) = S,$$

for $*$ = (ω) or $\{\omega\}$.

Proof. We assume, without loss of generality, that $\Omega = \mathbb{R}^n$; we construct the same distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ constructed in [3, Thm. 18] and follow the ideas therein and in [12, Thm. 8.1.4].

We choose a sequence $(x_k, \theta_k) \in S$ with $|\theta_k| = 1$ so that every $(x, \theta) \in S$ with $|\theta| = 1$ is the limit of a subsequence.

We set $\sigma(t) := \omega(t^{3/2})$ and separate the Beurling and the Roumieu cases.

Beurling case: Take $\phi \in \mathcal{D}_{(\omega)}(\mathbb{R}^n)$ with $\hat{\phi}(0) = 1$ and define, as in (158) of [3],

$$(3.29) \quad u(x) = \sum_{k=1}^{+\infty} e^{-\sigma(k^{d/m})} \phi(k(x - x_k)) e^{ik^3 \langle x, \theta_k \rangle},$$

where d is the constant of (2.4) with $0 < d < m$ since P is not elliptic by assumption. This is a continuous function in \mathbb{R}^n and it was already proved in [3, Thm. 18] that

$$\emptyset \neq \text{WF}_{(\omega)}^P(u) \subseteq S.$$

Let us now prove the other inclusion. Fix $(x_0, \xi_0) \in S$ and assume by contradiction that $(x_0, \xi_0) \notin \text{WF}_{(\omega)}^P(u)$. Then, by Corollary 3.14, there exist a neighborhood U of x_0 , an open conic neighborhood Γ of ξ_0 and $\psi \in \mathcal{D}_{\{\bar{\sigma}\}}(\mathbb{R}^n)$ with $\psi \equiv 1$ in U such that (3.24) is satisfied.

Set then

$$\phi_k(y) = \psi\left(\frac{y}{k} + x_k\right) \phi(y).$$

Since $\sigma(t) = o(\bar{\sigma}(t))$ by assumption, we have that $\mathcal{D}_{\{\bar{\sigma}\}}(\mathbb{R}^n) \subseteq \mathcal{D}_{(\sigma)}(\mathbb{R}^n)$ by [7, Prop. 4.7] and therefore $\{\phi_k\}_{k \in \mathbb{N}}$ is a bounded sequence in $\mathcal{D}_{(\sigma)}(\mathbb{R}^n)$, taking into account that $\text{supp } \phi_k \subseteq \text{supp } \phi$ for all $k \in \mathbb{N}$. By [7, Prop. 3.4], for each $h \in \mathbb{N}$ there is $C_h > 0$ such that

$$(3.30) \quad |\hat{\phi}_j(\xi)| \leq C_h e^{-h\sigma(\xi)} \quad \forall j \in \mathbb{N}, \xi \in \mathbb{R}^n.$$

Moreover, following [3], we have

$$\begin{aligned} \mathcal{F}(P(D)^N(\psi u))(\xi) &= \sum_{j=1}^{+\infty} e^{-\sigma(j^{d/m})} P(\xi)^N \mathcal{F}\left(\phi_j(j(x - x_j)) e^{ij^3 \langle x, \theta_j \rangle}\right) \\ &= \sum_{j=1}^{+\infty} e^{-\sigma(j^{d/m})} P(\xi)^N j^{-n} \hat{\phi}_j\left(\frac{\xi - j^3 \theta_j}{j}\right) e^{i \langle x_j, j^3 \theta_j - \xi \rangle}. \end{aligned}$$

If x_k is close to x_0 and k is large enough, then $\phi_k = \phi$ and

$$\begin{aligned} |\mathcal{F}(P(D)^N(\psi u))(k^3\theta_k)| &= \left| e^{-\sigma(k^{d/m})} k^{-n} P(k^3\theta_k)^N \right. \\ &\quad \left. + \sum_{j \neq k} e^{-\sigma(j^{d/m})} j^{-n} P(k^3\theta_k)^N \widehat{\phi}_j \left(\frac{k^3\theta_k - j^3\theta_j}{j} \right) e^{i\langle x_j, j^3\theta_j - k^3\theta_k \rangle} \right| \\ &\geq |P(k^3\theta_k)|^N \left(e^{-\sigma(k^{d/m})} k^{-n} - \sum_{j \neq k} e^{-\sigma(j^{d/m})} j^{-n} C_h e^{-h\sigma\left(\frac{k^3\theta_k - j^3\theta_j}{j}\right)} \right) \\ &\geq \delta^N k^{3Nd} (e^{-\sigma(k^{d/m})} k^{-n} - C'_h e^{-h\sigma(k)}), \end{aligned}$$

for some $C'_h > 0$, because of (3.30), (2.4) and

$$|k^3\theta_k - j^3\theta_j| \geq |k^3 - j^3| \geq k^2 + kj + j^2 \geq kj \quad \text{if } k \neq j.$$

But for every fixed $h \geq 2$ there exists $k_0 \in \mathbb{N}$ such that

$$\frac{\log C'_h}{\sigma(k)} + \frac{\log 2}{\sigma(k)} + \frac{n \log k}{\sigma(k)} + \frac{\sigma(k^{d/m})}{\sigma(k)} \leq h \quad \forall k \geq k_0$$

since $\log k = o(\sigma(k))$ and $0 < d < m$.

Therefore

$$(3.31) \quad |\mathcal{F}(P(D)^N(\psi u))(k^3\theta_k)| \geq \frac{1}{2} \delta^N k^{3Nd} k^{-n} e^{-\sigma(k^{d/m})} \quad \forall k \geq k_0.$$

On the other hand, by (3.24):

$$(3.32) \quad |\mathcal{F}(P(D)^N(\psi u))(k^3\theta_k)| = |P(k^3\theta_k)|^N |\widehat{\psi u}(k^3\theta_k)| \leq C_{h,\ell} e^{h\varphi^*\left(\frac{Nm}{h}\right)} (1 + |k^3\theta_k|)^{-\ell}.$$

But (3.31) and (3.32) give a contradiction for k large enough (see [3] for more details). Therefore $(x_0, \xi_0) \in \text{WF}_{(\omega)}^P(u)$ and $S \subseteq \text{WF}_{(\omega)}^P(u)$.

Roumieu case: Take $\phi \in \mathcal{D}_{\{\sigma\}}(\mathbb{R}^n)$ with $\widehat{\phi}(0) = 1$; choose, by Lemma 1.7 of [7], a non-quasianalytic weight function $\alpha(t)$ such that $\log t = o(\alpha(t))$ and $\alpha(t) = o(\sigma(t))$ for $t \rightarrow +\infty$, and define, as in (138) of [3],

$$u(x) = \sum_{k=1}^{+\infty} e^{-\frac{\sigma(k^{d/m})}{\alpha(k^{d/m})} \log k} \phi(k(x - x_k)) e^{ik^3 \langle x, \theta_k \rangle}.$$

This is a continuous function in \mathbb{R}^n and it was already proved in [3, Thm. 18] that

$$\emptyset \neq \text{WF}_{\{\omega\}}^P(u) \subseteq S.$$

Let us now prove the other inclusion. Fix $(x_0, \xi_0) \in S$ and assume by contradiction that $(x_0, \xi_0) \notin \text{WF}_{\{\omega\}}^P(u)$. Then, by Corollary 3.15, there exist a neighborhood U of x_0 , an open conic neighborhood Γ of ξ_0 and $\psi \in \mathcal{D}_{\{\bar{\sigma}\}}(\mathbb{R}^n)$ with $\psi \equiv 1$ in U such that (3.27) is satisfied.

Define, as in the Beurling case, $\phi_k(y) = \psi\left(\frac{y}{k} + x_k\right) \phi(y) \in \mathcal{D}_{\{\sigma\}}(\mathbb{R}^n)$ such that, by [7, Prop. 3.4], there exists $C, h > 0$:

$$|\widehat{\phi}_j(\xi)| \leq C e^{-\frac{1}{h}\sigma(\xi)} \quad \forall j \in \mathbb{N}, \xi \in \mathbb{R}^n.$$

Then, if x_k is close to x_0 and k is large enough,

$$\begin{aligned}
|\mathcal{F}(P(D)^N(\psi u))(k^3\theta_k)| &= \left| e^{-\frac{\sigma(k^{d/m})}{\alpha(k^{d/m})} \log k} k^{-n} P(k^3\theta_k)^N \right. \\
&\quad \left. + \sum_{j \neq k} e^{-\frac{\sigma(j^{d/m})}{\alpha(j^{d/m})} \log j} j^{-n} P(k^3\theta_k)^N \widehat{\phi_j} \left(\frac{k^3\theta_k - j^3\theta_j}{j} \right) e^{i\langle x_j, j^3\theta_j - k^3\theta_k \rangle} \right| \\
&\geq |P(k^3\theta_k)^N| \left(e^{-\frac{\sigma(k^{d/m})}{\alpha(k^{d/m})} \log k} k^{-n} \right. \\
&\quad \left. - \sum_{j \neq k} e^{-\frac{\sigma(j^{d/m})}{\alpha(j^{d/m})} \log j} j^{-n} C e^{-\frac{1}{h}\sigma\left(\frac{k^3\theta_k - j^3\theta_j}{j}\right)} \right) \\
&\geq \delta^N k^{3Nd} \left(e^{-\frac{\sigma(k^{d/m})}{\alpha(k^{d/m})} \log k} k^{-n} - C' e^{-\frac{1}{h}\sigma(k)} \right) \\
(3.33) \quad &\geq \frac{1}{2} \delta^N k^{3Nd} e^{-\frac{\sigma(k^{d/m})}{\alpha(k^{d/m})} \log k} k^{-n}
\end{aligned}$$

if k is large enough, because

$$\frac{\log(2C')}{\sigma(k)} + \frac{\sigma(k^{d/m})}{\sigma(k)} \frac{\log k}{\alpha(k^{d/m})} + \frac{n \log k}{\sigma(k)} \leq \frac{1}{h}$$

if k is sufficiently large, since $\log k = o(\alpha(k))$, $\log k = o(\sigma(k))$ and $0 < d < m$.

On the other hand, by (3.27) there exists $h \in \mathbb{N}$ such that

$$|\mathcal{F}(P(D)^N(\psi u))(k^3\theta_k)| = |P(k^3\theta_k)|^N |\widehat{\psi u}(k^3\theta_k)| \leq C_\ell e^{\frac{1}{h}\varphi^*(Nmh)} (1 + |k^3\theta_k|)^{-\ell}$$

which contradicts (3.33) (see [3] for more details).

Therefore $(x_0, \xi_0) \in \text{WF}_{\{\omega\}}^P(u)$ and $\text{WF}_{\{\omega\}}^P(u) = S$. □

Example 3.17. Let Ω be an open subset of \mathbb{R}^n and $P(D)$ a linear partial differential operator with constant coefficients and of order m . By Theorems 4.1 and 4.8 of [1], for $*$ = (ω) or $\{\omega\}$, we have

$$(3.34) \quad \text{WF}_*(u) \subseteq \text{WF}_*(Pu) \cup \Sigma, \quad u \in \mathcal{D}'(\Omega),$$

where

$$\Sigma := \{(x, \xi) \in \Omega \times \mathbb{R}^n \setminus \{0\} : P_m(\xi) = 0\}$$

is the characteristic set of P (here P_m is the principal part of P).

This implies that if u is a solution of $P(D)u = f$, then

$$(3.35) \quad \text{WF}_*(u) \subseteq \Sigma, \quad \text{for each } f \in \mathcal{E}_*(\Omega),$$

since $\text{WF}_*(f) = \emptyset$ for $f \in \mathcal{E}_*(\Omega)$.

Now, by Theorem 13 of [3]

$$\text{WF}_*(u) \subseteq \text{WF}_*^P(u) \cup \Sigma, \quad u \in \mathcal{D}'(\Omega),$$

so that we can improve (3.35) saying that

$$\text{WF}_*(u) \subseteq \Sigma \quad \text{if } f \in \mathcal{E}_*^P(\Omega).$$

Indeed, if $f \in \mathcal{E}_*^P(\Omega) \setminus \mathcal{E}_*(\Omega)$ then $\text{WF}_*^P(f) = \emptyset$ and hence

$$\text{WF}_*(u) \subseteq \text{WF}_*^P(u) \cup \Sigma = \text{WF}_*^P(Pu) \cup \Sigma = \text{WF}_*^P(f) \cup \Sigma = \Sigma.$$

Let us now prove a result that establishes the relationship between the wave front set in the Beurling class and in the Roumieu class. See also [1, Proposition 4.5] and [9, Proposition 2] for similar results in this setting for the usual wave front set, even for quasianalytic weight functions.

Proposition 3.18. *Let $P(D)$ be a hypoelliptic linear partial differential operator of order m with constant coefficients, Ω an open subset of \mathbb{R}^n and $u \in \mathcal{D}'(\Omega)$. Let σ_0 and ω be two non-quasianalytic weight functions such that, for $t \rightarrow +\infty$, $\sigma_0(t) = o(\omega(t))$ and $\omega(t^\gamma) = o(t^{1/s})$ for γ as in (2.5) and $s > 1$. Then*

$$\text{WF}_{\{\omega\}}^P(u) = \overline{\bigcup_{\sigma \in S} \text{WF}_{(\sigma)}^P(u)},$$

where $S := \{\sigma \text{ non-quasianalytic weight function: } \sigma_0 \leq \sigma = o(\omega)\}$.

Proof. Let us first prove that

$$(3.36) \quad \overline{\bigcup_{\sigma \in S} \text{WF}_{(\sigma)}^P(u)} \subseteq \text{WF}_{\{\omega\}}^P(u).$$

To do this, we fix $(x_0, \xi_0) \notin \text{WF}_{\{\omega\}}^P(u)$. By Corollary 3.15 there exist a neighborhood U of x_0 , an open conic neighborhood Γ of ξ_0 and $\psi \in \mathcal{D}_{\{t^{1/s}\}}(\mathbb{R}^n)$ with $\psi \equiv 1$ in U such that (3.26) is satisfied for some $k_0 \in \mathbb{N}$ and $C > 0$.

If $\sigma(t) = o(\omega(t))$, then for every $k \in \mathbb{N}$ there exists $t_k > 0$ such that

$$k\sigma(t) \leq \frac{1}{k_0}\omega(t) \quad \forall t \geq t_k.$$

Since P is hypoelliptic there exists then $R_k > 0$ such that

$$k\sigma(|P(\xi)|^{\frac{1}{m}}) \leq \frac{1}{k_0}\omega(|P(\xi)|^{\frac{1}{m}}) \quad \forall |\xi| \geq R_k,$$

and therefore there exists $C_k > 0$ such that, by (3.26),

$$|\widehat{\psi u}(\xi)| \leq C e^{-\frac{1}{k_0}\omega(|P(\xi)|^{\frac{1}{m}})} \leq C_k e^{-k\sigma(|P(\xi)|^{\frac{1}{m}})} \quad \forall \xi \in \mathbb{R}^n.$$

This proves, by Corollary 3.14, that $(x_0, \xi_0) \notin \text{WF}_{(\sigma)}^P(u)$ and hence

$$\bigcup_{\sigma \in S} \text{WF}_{(\sigma)}^P(u) \subseteq \text{WF}_{\{\omega\}}^P(u).$$

Since the wave front set $\text{WF}_{\{\omega\}}^P(u)$ is always a closed set, we have the inclusion (3.36).

Let us prove the other inclusion. Take $(x_0, \xi_0) \notin \overline{\bigcup_{\sigma \in S} \text{WF}_{(\sigma)}^P(u)}$. Then there exist a compact neighborhood K of x_0 and a closed conic neighborhood F of ξ_0 such that

$$(K \times F) \cap \overline{\bigcup_{\sigma \in S} \text{WF}_{(\sigma)}^P(u)} = \emptyset.$$

Take $\chi_N \in \mathcal{D}(K)$ with $\chi_N \equiv 1$ in a neighborhood $K' \subset K$ of x_0 which satisfies (3.21). Take then $\psi \in \mathcal{D}_{\{t^{1/s}\}}(K')$. By Proposition 3.12

$$\text{WF}_{(\sigma)}^P(\psi u) \subseteq \text{WF}_{(\sigma)}^P(u) \quad \forall \sigma \in S$$

and hence

$$(K \times F) \cap \text{WF}_{(\sigma)}^P(\psi u) = \emptyset \quad \forall \sigma \in S.$$

Consider then $g_N := \chi_{Nmp} P(D)^N \psi u$, for p sufficiently large so that, by [3, Cor. 10]:

$$\begin{aligned} \forall \sigma \in S, \forall k, \ell \in \mathbb{N} \exists C_{k,\ell,\sigma} > 0 \text{ s.t.} \\ |\widehat{g_N}(\xi)| \leq C_{k,\ell,\sigma} e^{k\varphi_\sigma^*(\frac{Nm}{k})} (1 + |\xi|)^{-\ell} \quad \forall \xi \in F, \end{aligned}$$

where φ_σ^* is the Young conjugate of $\varphi_\sigma(t) = \sigma(e^t)$.

But

$$g_N = \chi_{Nmp} P(D)^N \psi u = P(D)^N \psi u \quad \text{in } \mathbb{R}^n$$

since $\chi_{Nmp} \equiv 1$ on $\text{supp } \psi$. Therefore

$$|P(\xi)^N \widehat{\psi u}(\xi)| = |\widehat{g_N}(\xi)| \leq C_{k,\ell,\sigma} e^{k\varphi_\sigma^*(\frac{Nm}{k})} (1 + |\xi|)^{-\ell} \quad \forall \xi \in F,$$

which implies, by Lemma 3.5:

$$(3.37) \quad |\widehat{\psi u}(\xi)| \leq C_{k,\sigma} e^{-k\sigma(|P(\xi)|^{1/m})} \quad \forall \xi \in F.$$

We want to prove that there exists $\bar{k} \in \mathbb{N}$ such that (3.26) is satisfied on F (this would imply, by Corollary 3.15, that $(x_0, \xi_0) \notin \text{WF}_{\{\omega\}}^P(u)$). We argue by contradiction and assume that for every $n \in \mathbb{N}$ there exists $\xi_n \in F$ such that

$$|P(\xi_n)| \geq \delta |\xi_n|^d \rightarrow +\infty$$

and

$$(3.38) \quad |\widehat{\psi u}(\xi_n)| \geq n e^{-\frac{1}{n}\omega(|P(\xi_n)|^{1/m})} \quad \forall n \in \mathbb{N}.$$

Since $\sigma(t) = o(\omega(t))$, for $\sigma \in S$, for every $n \in \mathbb{N}$ there exists $k_n \geq n$ such that

$$\frac{\sigma(|P(\xi_k)|^{1/m})}{\omega(|P(\xi_k)|^{1/m})} < \frac{1}{n} \quad \forall k \geq k_n.$$

This would imply, together with (3.38), that

$$|\widehat{\psi u}(\xi_{k_n})| \geq k_n e^{-\frac{1}{k_n}\omega(|P(\xi_{k_n})|^{1/m})} \geq n e^{-\frac{1}{n}\omega(|P(\xi_{k_n})|^{1/m})} > n e^{-\sigma(|P(\xi_{k_n})|^{1/m})},$$

contradicting (3.37) for $k = 1$.

Therefore $(x_0, \xi_0) \notin \text{WF}_{\{\omega\}}^P(u)$ and the proposition is proved. \square

4. Operators with constant strength

We give some applications to linear partial differential operators with variable coefficients with constant strength. We recall, from [13], the following:

Definition 4.1. Let $P(D)$ and $Q(D)$ be two linear partial differential operators with constant coefficients. We say that P is weaker than Q , and write $P \prec Q$, if there exists a constant $C > 0$ such that

$$\tilde{P}(\xi) \leq C \tilde{Q}(\xi) \quad \forall \xi \in \mathbb{R}^n,$$

where $\tilde{P}(\xi) := \sqrt{\sum_\alpha |D^\alpha P(\xi)|^2}$. We say that P and Q are equally strong if there exists a constant $C > 0$ such that

$$C^{-1} \tilde{P}(\xi) \leq \tilde{Q}(\xi) \leq C \tilde{P}(\xi) \quad \forall \xi \in \mathbb{R}^n.$$

Remark 4.2. If $P(D)$ and $Q(D)$ are equally strong and hypoelliptic, then by Theorem 2.4 it follows that there are two constants $C, C' > 0$ such that

$$\begin{aligned} |P(\xi)|^2 &\leq C(1 + |Q(\xi)|^2) & \forall \xi \in \mathbb{R}^n \\ |Q(\xi)|^2 &\leq C'(1 + |P(\xi)|^2) & \forall \xi \in \mathbb{R}^n. \end{aligned}$$

In particular, $\deg P = \deg Q$.

We recall from [13, Thm. 11.1.9] the following

Theorem 4.3. If $P(D)$ and $Q(D)$ are equally strong and $P(D)$ is hypoelliptic, then also $Q(D)$ is hypoelliptic. Moreover, if $d_P(\xi)$ and $d_Q(\xi)$ are the distance from $\xi \in \mathbb{R}^n$ to $V(P)$ and $V(Q)$ respectively, there exists then a constant $C > 0$ such that

$$(4.1) \quad C^{-1} \leq \frac{d_P(\xi) + 1}{d_Q(\xi) + 1} \leq C, \quad \forall \xi \in \mathbb{R}^n.$$

Lemma 4.4. *Let $P(D)$ and $Q(D)$ be two equally strong linear partial differential operators. Assume that P (and hence Q) is hypoelliptic and of order m . Let c_P and c_Q be the constants defined in (2.2) for P and Q respectively. Then $c_P = c_Q$.*

Proof. Let us first remark that condition (4.1) is equivalent to

$$(4.2) \quad \tilde{C}^{-1} \leq \frac{d_P(\xi)}{d_Q(\xi)} \leq \tilde{C}, \quad |\xi| \gg 1,$$

for some $\tilde{C} > 0$, because of condition (6) of Theorem 2.4.

Then (2.2) implies, for some $C, \bar{C} > 0$,

$$\begin{aligned} |\xi|^{c_P} \leq C^{-1} d_P(\xi) \leq C^{-1} \tilde{C} d_Q(\xi) &\Rightarrow c_Q \geq c_P \\ |\xi|^{c_Q} \leq \bar{C}^{-1} d_Q(\xi) \leq \bar{C}^{-1} \tilde{C} d_P(\xi) &\Rightarrow c_P \geq c_Q \end{aligned}$$

and $c_P = c_Q$. □

Remark 4.5. If $P(D)$ and $Q(D)$ are equally strong, hypoelliptic and of order m , and if γ_P and γ_Q are the constants defined in (2.5) for P and Q respectively, then (see Remark 2.5):

$$(4.3) \quad m \leq \gamma_P, \gamma_Q \leq \frac{m}{c_P} = \frac{m}{c_Q}.$$

Theorem 4.6. *Let $P(D)$ and $Q(D)$ be two equally strong linear partial differential operators and let ω be a non-quasianalytic weight function. If P is $*$ -hypoelliptic then also Q is $*$ -hypoelliptic, for $*$ = (ω) or $\{\omega\}$.*

Proof. From Remark 2.10 we have that P is hypoelliptic. Therefore also Q is hypoelliptic, by Theorem 4.3, and (4.2) is satisfied.

Beurling case. If P is (ω) -hypoelliptic, then by Theorem 2.7

$$\frac{\omega(\xi)}{d_Q(\xi)} \leq \tilde{C} \frac{\omega(\xi)}{d_P(\xi)} \rightarrow 0 \quad \text{as } |\xi| \rightarrow +\infty$$

and hence also Q is (ω) -hypoelliptic.

Roumieu case. If P is $\{\omega\}$ -hypoelliptic, then by Theorem 2.9

$$\omega(\xi) \leq C d_P(\xi) \leq C \tilde{C} d_Q(\xi), \quad |\xi| \gg 1$$

and hence also Q is $\{\omega\}$ -hypoelliptic. □

Now, from Theorems 3.3 and 3.4, and Corollaries 3.14 and 3.15 it is easy to deduce the following:

Proposition 4.7. *Let $P(D)$ and $Q(D)$ be two equally strong linear partial differential operators. Assume that P (and hence Q) is hypoelliptic and of order m . Let ω be a non-quasianalytic weight function such that $\omega(t^\gamma) = o(\sigma(t))$, as $t \rightarrow +\infty$, where $\gamma := \frac{m}{c_P} = \frac{m}{c_Q}$ as in (4.3) and $\sigma(t) = t^{1/s}$ for some $s > 1$. Let Ω be an open subset of \mathbb{R}^n and $u \in \mathcal{D}'(\Omega)$. Then*

$$(4.4) \quad \mathcal{E}_*^P(\Omega) = \mathcal{E}_*^Q(\Omega)$$

$$(4.5) \quad \text{WF}_*^P(u) = \text{WF}_*^Q(u)$$

for $*$ = (ω) or $\{\omega\}$.

Let us now consider linear partial differential operators $P(x, D)$ with variable coefficients on an open subset Ω of \mathbb{R}^n . We recall from [13] the following:

Definition 4.8. *A linear partial differential operator $P(x, D)$, with C^∞ coefficients on an open subset Ω of \mathbb{R}^n , is said to have constant strength in Ω if, for every $x_0, y_0 \in \Omega$, the differential operators with constant coefficients $P(x_0, D)$ and $P(y_0, D)$ are equally strong.*

The following result is due to Hörmander and Taylor (cf. Theorems 13.4.1, 13.4.2 and 13.4.4 of [13]):

Theorem 4.9. *Let Ω be an open subset of \mathbb{R}^n and $P(x, D)$ a linear partial differential operator with coefficients in $C^\infty(\Omega)$. Assume that $P(x, D)$ has constant strength. Then the following conditions are equivalent:*

(1) $P(x, D)$ is hypoelliptic in Ω , i.e.

$$\text{sing supp } u = \text{sing supp } P(\cdot, D)u, \quad u \in \mathcal{D}'(\Omega);$$

(2) $P(x_0, D)$ is hypoelliptic for some $x_0 \in \Omega$;

(3) $P(x_0, D)$ is hypoelliptic for all $x_0 \in \Omega$.

Moreover, if one of the above equivalent conditions is satisfied, then $P(x, D)$ is micro-hypoelliptic in Ω , i.e.

$$\text{WF}(u) = \text{WF}(P(\cdot, D)u), \quad u \in \mathcal{D}'(\Omega).$$

In the case of ω -hypoellipticity, from Theorem 4.6 we immediately obtain that for a linear partial differential operator with C^∞ coefficients $P(x, D)$ with constant strength, $P(x_0, D)$ is $*$ -hypoelliptic for some $x_0 \in \Omega$ if and only if $P(x_0, D)$ is $*$ -hypoelliptic for all $x_0 \in \Omega$, where $*$ = (ω) or $\{\omega\}$.

As a consequence of Theorem 4.9 and Lemma 4.4 we have that if $P(x, D)$ is a hypoelliptic linear partial differential operator of order m with C^∞ coefficients and of constant strength in an open subset Ω of \mathbb{R}^n , there is then a unique constant $c_P \in (0, 1] \cap \mathbb{Q}$ satisfying (2.2), in the sense that $c_{P(x_0, D)} = c_{P(x_1, D)} =: c_P$ for all $x_0, x_1 \in \Omega$. We can then uniquely define

$$(4.6) \quad \gamma_P := \frac{m}{c_P}.$$

Corollary 4.10. *Let Ω be an open subset of \mathbb{R}^n and $u \in \mathcal{D}'(\Omega)$. Let $P(x, D)$ be a linear partial differential operator with coefficients in $C^\infty(\Omega)$. Assume that $P(x, D)$ has constant strength in Ω and that $P(x_0, D)$ is hypoelliptic for some $x_0 \in \Omega$. Let ω be a non-quasianalytic weight function such that $\omega(t^\gamma) = o(\sigma(t))$, as $t \rightarrow +\infty$, where $\gamma = \gamma_P$ is the constant defined in (4.6) and $\sigma(t) = t^{1/s}$ for some $s > 1$. Then*

$$\begin{aligned} \mathcal{E}_*^{P(x, D)}(\Omega) &= \mathcal{E}_*^{P(x', D)}(\Omega) \quad \forall x, x' \in \Omega \\ \text{WF}_*^{P(x, D)}(u) &= \text{WF}_*^{P(x', D)}(u) \quad \forall x, x' \in \Omega \end{aligned}$$

for $*$ = (ω) or $\{\omega\}$.

Proof. It follows from Theorem 4.9 and Proposition 4.7. □

Corollary 4.11. *Let Ω be an open subset of \mathbb{R}^n and $u \in \mathcal{D}'(\Omega)$. Let $P(x, D)$ be a linear partial differential operator with coefficients in $C^\infty(\Omega)$. Assume that $P(x, D)$ has constant strength in Ω and that $P(x_0, D)$ is hypoelliptic for some $x_0 \in \Omega$. Let ω be a non-quasianalytic weight function such that $\omega(t^\gamma) = o(\sigma(t))$, as $t \rightarrow +\infty$, where $\gamma = \gamma_P$ is the constant defined in (4.6) and $\sigma(t) = t^{1/s}$ for some $s > 1$. Then*

$$\text{WF}_*(u) \subseteq \text{WF}_*^{P(x_1, D)}(u) \cup \left(\bigcap_{x' \in \Omega} \Sigma_{x'} \right) \quad \forall x_1 \in \Omega,$$

where $*$ = (ω) or $\{\omega\}$ and

$$\Sigma_{x'} := \{(x, \xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\}) : P_{m(x')}(x', \xi) = 0\}$$

with $P_{m(x')}(x', \xi)$ the principal part of $P(x', \xi)$.

Proof. By Theorem 13 of [3] we have that

$$\begin{aligned} \text{WF}_*(u) &\subseteq \text{WF}_*^{P(x', D)}(u) \cup \Sigma_{x'} \quad \forall x' \in \Omega \\ &= \text{WF}_*^{P(x_1, D)}(u) \cup \Sigma_{x'} \quad \forall x_1, x' \in \Omega \end{aligned}$$

by Corollary 4.10. This proves the thesis. □

Corollary 4.12. *Let Ω be an open subset of \mathbb{R}^n and $u \in \mathcal{D}'(\Omega)$. Let $P(x, D)$ be a linear partial differential operator with coefficients in $C^\infty(\Omega)$. Assume that $P(x, D)$ has constant strength in Ω and that $P(x_0, D)$ is elliptic for some $x_0 \in \Omega$. Let ω be a non-quasianalytic weight function such that $\omega(t^\gamma) = o(\sigma(t))$, as $t \rightarrow +\infty$, where $\gamma = \gamma_P$ is the constant defined in (4.6) and $\sigma(t) = t^{1/s}$ for some $s > 1$.*

Then $P(x_1, D)$ is $$ -micro-hypoelliptic for all $x_1 \in \Omega$, i.e.*

$$\mathrm{WF}_*(u) = \mathrm{WF}_*(P(x_1, D)u) \quad \forall x_1 \in \Omega,$$

for $*$ = (ω) or $\{\omega\}$.

Proof. By [3, Rem. 14] we have that

$$\mathrm{WF}_*(u) = \mathrm{WF}_*^{P(x_0, D)}(u)$$

by the ellipticity of $P(x_0, D)$.

Therefore, by Corollary 4.10 and [3, Rem. 12 and Prop. 9], for all $x_1 \in \Omega$:

$$\begin{aligned} \mathrm{WF}_*(u) &= \mathrm{WF}_*^{P(x_0, D)}(u) = \mathrm{WF}_*^{P(x_1, D)}(u) \\ &= \mathrm{WF}_*^{P(x_1, D)}(P(x_1, D)u) \subseteq \mathrm{WF}_*(P(x_1, D)u) \subseteq \mathrm{WF}_*(u), \end{aligned}$$

and hence the thesis. \square

The following corollary can be proved directly with the results obtained in [9], but we present it here as a consequence of the previous theorems.

Corollary 4.13. *Let Ω be an open subset of \mathbb{R}^n and $u \in \mathcal{D}'(\Omega)$. Let $P(x, D)$ be a linear partial differential operator with coefficients in $\mathcal{E}_{\{\sigma\}}(\Omega)$. Assume that $P(x, D)$ has constant strength in Ω and that $P(x_0, D)$ is elliptic for some $x_0 \in \Omega$. Let ω be a non-quasianalytic weight function such that $\omega(t^\gamma) = o(t^{1/s})$, for $\gamma = \gamma_P$ as in (4.6) and $s > 1$, and such that $\omega(t) = o(\sigma(t))$ for a non-quasianalytic weight function σ satisfying, for some $c > 0$,*

$$(4.7) \quad \int_1^{+\infty} \frac{\sigma(ty)}{t^2} dt \leq c\sigma(t) + c \quad y > 0.$$

Then

$$(4.8) \quad \mathrm{WF}_*(u) = \mathrm{WF}_*(P(\cdot, D)u) = \mathrm{WF}_*(P(x_1, D)u) \quad \forall x_1 \in \Omega,$$

for $*$ = (ω) or $\{\omega\}$.

Proof. By Theorem 3 of [9] we have that

$$\mathrm{WF}_*(u) = \mathrm{WF}_*(P(\cdot, D)u).$$

By Corollary 4.12 we have

$$\mathrm{WF}_*(u) = \mathrm{WF}_*(P(x_1, D)u) \quad \forall x_1 \in \Omega.$$

Therefore also

$$\mathrm{WF}_*(P(\cdot, D)u) = \mathrm{WF}_*(P(x_1, D)u) \quad \forall x_1 \in \Omega$$

is valid. \square

Example 4.14. Let $q(\xi) \geq 0$ be a hypoelliptic polynomial, $h, m, m' \in \mathbb{N}_0$, $m > m'$, and consider the operator $P(x, D)$ with symbol

$$p(x, \xi) := |x|^{2h} q(\xi)^m + q(\xi)^{m'}.$$

It was proved in [8, Ex. 3.6] that there exist $0 \leq \delta < \rho \leq 1$ such that, for $a < d := \rho - \delta < 1$, $a/d < r < 1$, $\sigma(t) = t^r$ and $\omega(t) = o(t^a)$:

$$(4.9) \quad |p(x, \xi)| \geq ce^{-\omega(\xi)} \quad \forall x \in \mathbb{R}^n, |\xi| \gg 1$$

for some $c > 0$, and

$$(4.10) \quad |D_x^\alpha D_\xi^\beta p(x, \xi)| \leq C^{|\alpha|+|\beta|} \beta! e^{\frac{1}{k} \varphi_\sigma^*(|\alpha|k)} |p(x, \xi)| (1 + |\xi|)^{-\rho|\beta|+\delta|\alpha|} \quad \forall x \in \mathbb{R}^n, |\xi| \gg 1$$

for some $C > 0$ and $k \in \mathbb{N}$.

By [8, Thm. 3.4] and [9, Thm. 2] we have

$$\text{WF}_{(\omega)}(u) = \text{WF}_{(\omega)}(P(\cdot, D)u), \quad u \in \mathcal{E}'_{(\omega)}(\Omega).$$

Analogously, since (4.9) and (4.10) hold also for every fixed $x_1 \in \Omega$, we have that

$$\text{WF}_{(\omega)}(u) = \text{WF}_{(\omega)}(P(x_1, D)u), \quad \forall x_1 \in \Omega,$$

and hence (4.8) holds for all $u \in \mathcal{E}'_{(\omega)}(\Omega)$, and hence for all $u \in \mathcal{E}'(\Omega)$.

Note that P is not of constant strength, and that P is elliptic in some $x_0 \in \Omega$ if and only if $q(\xi)$ is elliptic.

Example 4.15. Let us consider, for $m \geq 1$, the operator

$$P(x, y, D_x, D_y) = D_x^{2m} + x^{2h} D_y^{2m+2}.$$

It was proved in [24, Thm. 3.1] that if $h > m$ and $\omega(t) = t^{1/s}$, with $s \geq 1 + \frac{1}{m}$, then

$$(4.11) \quad \text{WF}_{\{\omega\}}(u) = \text{WF}_{\{\omega\}}(P(\cdot, D)u), \quad \forall u \in \mathcal{D}'(\mathbb{R}^2).$$

However, for fixed $(x_1, y_1) = (0, y_1)$ the operator $P(0, y_1, D_x, D_y) = D_x^{2m}$ is not $\{\omega\}$ -hypoelliptic, since every $f(x, y) = f(y) \in \mathcal{D}'(\mathbb{R}) \setminus \mathcal{E}_{\{\omega\}}(\mathbb{R})$ solves $P(0, y_1, D_x, D_y)f = 0$, and hence it is not $\{\omega\}$ -microhypoelliptic by [3, Cor. 11]:

$$\text{WF}_{\{\omega\}}(P(0, y_1, D_x, D_y)f) \not\subseteq \text{WF}_{\{\omega\}}(f).$$

In particular, by (4.11):

$$\text{WF}_{\{\omega\}}(P(0, y_1, D_x, D_y)f) \not\subseteq \text{WF}_{\{\omega\}}(P(\cdot, D)f).$$

Note that P does not have constant strength and $P(x_0, y_0, D_x, D_y)$ is not elliptic for any $(x_0, y_0) \in \mathbb{R}^2$.

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